CHAPTER 5:

SMOOTH SUBMANIFOLDS

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Smooth Submanifolds

As usual, a sub*thing* is a subset $S \subseteq T$ of a *thing* T such that S can be considered a *thing* in its own right.
We've already seen that an open subset of a smooth manifold can be viewed as a smooth manifold.

The first kind of submanifolds we consider are the following.

Definition (Embedded Submanifold)

Let M be a manifold (without boundary). An *embedded submanifold* is a subset $S \subseteq M$ that is a manifold in the subspace topology, endowed with a smooth structure s.t. the inclusion map $S \hookrightarrow M$ is a smooth embedding.

Other books might use the term "regular submanifolds".

Embedded Submanifolds

If S is an embedded submanifold of M, the *codimension of S in M* is the difference

$$\dim M - \dim S.$$

M is called the *ambient manifold* of S.

The easiest submanifolds to characterise are those of codimension zero.

Proposition 5.1 (Open Submanifolds)

Let M be a smooth manifold, and $S \subseteq M$ be an embedded submanifold. Then

$$S \textit{ has codimension } 0 \iff S \textit{ is an open submanifold.}$$

Embedded Submanifolds

Proposition 5.1 (Open Submanifolds)

S has codimension $0 \iff S$ is an open submanifold.

Proof of 5.1.

Let $U \subseteq M$ be an open submanifold, and let $\iota\colon U \hookrightarrow M$ be the inclusion map. We've already seen that U is a smooth manifold of the same dimension as M, so its codimension is zero. Now ι is a smooth immersion, and because U has the subspace topology, ι is a smooth embedding, making it an embedded submanifold.

Conversely, if U is any codimension 0 embedded submanifold of M, the inclusion ι is a smooth embedding (by definition), and therefore it is a local diffeomorphism (proposition 4.8) and an open map (proposition 4.6). Thus, U is an open subset of M. $\square$

Embedded Submanifolds

The following results demonstrate other ways to obtain embedded submanifolds.

Proposition 5.2 (Images of Embeddings)

If M and N are smooth manifolds and $F\colon N \to M$ is a smooth embedding, then $F(N)$ is an embedded submanifold of M (with the subspace topology).

Corollary 5.3 (Slices of Product Manifolds)

*Let M and N be smooth manifolds. For each $p \in N$, the **slice** $M \times \{p\}$ is an embedded submanifold of $M \times N$ diffeomorphic to M.*

Proof.

Use proposition 5.1 with $F\colon M \to M \times N$ s.t. $F(x) = (x, p)$. $\square$

Embedded Submanifolds

For a function $f\colon X \to Y$, let Γf denote its graph

$$\Gamma f := \{(x, y) \in X \times Y : f(x) = y\}.$$

Proposition 5.4 (Graphs)

Let M, N be smooth manifolds of dimension m, n (resp.), let $U \subseteq M$ be open, and let $f\colon U \to N$ be a smooth map. Then Γf is an embedded submanifold of $M \times N$.

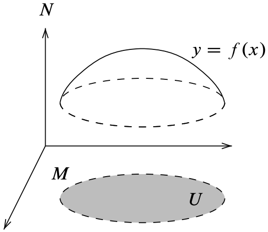

Figure: A graph as an embedded submanifold

Embedded Submanifold

For some purposes, being an embedded submanifold is not a strong enough condition. We say an embedded submanifold $S \subseteq M$ is *properly embedded* if the inclusion $S \hookrightarrow M$ is a proper map.

Proposition 5.5

Let S be an embedded submanifold of a smooth manifold M. Then

$$S \text{ properly embedded} \iff S \text{ is a closed subset of } M.$$

Proof.

If S is properly embedded, then it is closed (theorem A.57). Conversely, if S is closed in M, then by proposition A.53(c), the inclusion $S \hookrightarrow M$ is proper. □

Slice Charts for Embedded Submanifolds

In the next theorem, we will show that we can model embedded submanifolds locally using the standard embedding of $\mathbb{R}^k$ into $\mathbb{R}^n$, where we identify

$$\mathbb{R}^k \cong \{(x^1, \ldots, x^k, x^{k+1}, \ldots, x^n) : x^{k+1} = \cdots = x^n = 0\} \subseteq \mathbb{R}^n.$$

Definition (k-slice)

Let U be an open subset of $\mathbb{R}^n$, and $0 \leqslant k \leqslant n$. A *$k$-dmiensional slice of U* (or a k-slice) is any subset of the form

$$\{(x^1, \ldots, x^k, x^{k+1}, \ldots, x^n) : x^\ell = c^\ell \text{ for } 1 \leqslant \ell \leqslant n\},$$

where c^ℓ are constants for $1 \leqslant \ell \leqslant n$.

For smooth charts (U, φ) on a smooth manifold M, we also say that $S \subseteq U$ is a k-slice of U if $\varphi(S)$ is a k-slice of $\varphi(U)$.

Slice Charts for Embedded Submanifolds

Given a subset $S \subseteq M$ and an integer $k \geqslant 0$, we say that S satisfies the *local k-slice condition* if each point of S is in the domain of some smooth chart (U, φ) for M s.t. $S \cap U$ is a single k-slice in U.

Any such chart is called a *slice chart for S in M*, and the corresponding coordinates $(x^1, \ldots, x^n)$ are called *slice coordinates*.

Theorem 5.8 (Local Slice Criterion for Embedded Submanifolds)

Let M be a smooth n-manifold. If $S \subseteq M$ is an embedded k-dimensional submanifold, then S satisfies the local k-slice condition.

Conversely, any subset $S \subseteq M$ satisfying the k-slice condition is a topological manifold of dimension k (with the subspace topology), and has a smooth structure making it a k-dimensional embedded submanifold.

Slice Charts for Embedded Submanifolds

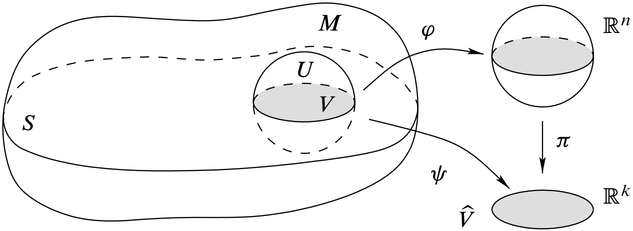

Figure: A chart for a subset satisfying the k-slice condition

Slice Charts for Embedded Submanifolds

Example 5.9 (Spheres)

For any $n \geqslant 0$, the sphere $\mathbb{S}^n$ is an embedded submanifold of $\mathbb{R}^{n+1}$, because $\mathbb{S}^n \cap \{x : x^i > 0\}$ is locally the graph of

$$f(u) = \sqrt{1 - |u|^2}$$

(example 1.4), and similarly the other part of $\mathbb{S}^n$ is the graph of $-f$.

Since every point in $\mathbb{S}^n$ is in one of these sets, $\mathbb{S}^n$ satisfies the local n-slice condition and is therefore an embedded submanifold of $\mathbb{R}^{n+1}$. The smooth structure that is induced on $\mathbb{S}^{n+1}$ is the same as the one defined in chapter 1. In fact, the coordinates for $\mathbb{S}^n$ determined by these slice charts are exactly the graph coordinates defined in example 1.31.

Level Sets

Embedded submanifolds are usually presented as solution sets of (systems of) equations.

Definition (Level set)

If $\Phi\colon M \to N$ is any map and c is any point on N, we call $\Phi^{-1}(c)$ a *level set of* Φ.

In the case that $N = \mathbb{R}^k$ and $c = 0$, $\Phi^{-1}(0)$ is called the zero set of Φ.

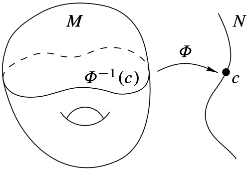

Figure: Level set

Not all level sets of smooth functions are smooth submanifolds. For instance, take the three smooth functions

$$\Phi_1(x, y) \underset{\checkmark}{:=} x^2 - y \qquad \Phi_2(x, y) \underset{\text{✗}}{:=} x^2 - y^2 \qquad \Phi_3(x, y) \underset{\text{✗}}{:=} x^2 - y^3$$

Level Sets

Theorem 5.12 (Constant-Rank Level Set Theorem)
Let M and N be smooth manifolds, and $\Phi\colon M \to N$ be a smooth map with constant rank r. Each level set of Φ is a properly embedded submanifold of codimension r in M.

Proof.
Let $k = \dim M - r$, let $c \in N$ be arbitrary, and let $S = \Phi^{-1}(c) \subseteq M$. By the rank theorem, for each $p \in S$, there are smooth charts (U, φ) centred at p and (V, ψ) centred at $c = \Phi(p)$ in which Φ has a coordinate representation $(x^1, \ldots, x^r, 0, \ldots, 0)$, and therefore $S \cap U$ is the slice $\{(x^1, \ldots, x^r, x^{r+1}, \ldots, x^{\dim M}) \in U : x^1 = \cdots x^r = 0\}$. Thus S satisfies the local k-slice condition, so it is an embedded submanifold with dimension k. It is closed in M by continuity, so by proposition 5.5, it is properly embedded. $\square$

Level Sets

The following result corresponds to the result of linear algebra that a surjective linear map $\Lambda\colon \mathbb{R}^m \to \mathbb{R}^r$ has kernel with codimension r.

In the context of smooth manifolds, surjective linear maps become smooth submersions.

Corollary 5.13 (Submersion Level Set Theorem)

If M and N are smooth manifolds and $\Phi\colon M \to N$ is a smooth submersion, then each level set of Φ is a properly embedded submanifold whose codimension is equal to the dimension of N.

Proof.

Every smooth submersion has constant rank equal to the dimension of its codomain. □

Immersed Submanifolds

These are more general than embedded submanifolds. They are important for chapter 7 when considering Lie subgroups.

Definition (Immersed Submanifold)

An immersed submanifold of M is a subset $S \subseteq M$ that is a manifold in a topology (not necessarily the subspace topology), endowed with a smooth structure such that $S \hookrightarrow M$ is a smooth immersion.

Analogously, we have the notion of codimension, and that they are they are images of immersions.

The Tangent Space to a Submanifold

If S is a smooth submanifold of $\mathbb{R}^n$, we intuitively think of the tangent space T_pS at $p \in S$ as a subspace of the tangent space $T_p\mathbb{R}^n$. Similarly, the tangent space to a smooth submanifold of an abstract smooth manifold can be viewed as a subspace of the tangent space to the ambient manifold, once we make appropriate identifications.

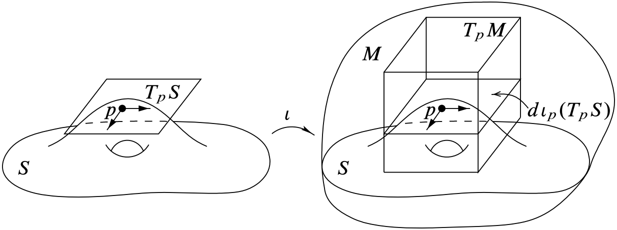

Figure: The tangent space to an embedded submanifold

The Tangent Space to a Submanifold

Let M be a smooth manifold, and let $S \subseteq M$ be an immersed (or embedded) submanifold. Since the inclusion $\iota\colon S \hookrightarrow M$ is a smooth immersion, at each point $p \in S$, we have an injective linear map $d\iota_p\colon T_pS \to T_pM$.

In terms of derivations, this injection works as follows. For any $v \in T_pS$, the image $\tilde{v} = d\iota_p(v) \in T_pM$ acts on smooth functions M by

$$\tilde{v}f = d\iota_p(v)f = v(f \circ \iota) = v(f \restriction S).$$

We adopt the convention of identifying T_pS with its image under this map, thereby thinking of T_pS as a linear subspace of T_pM as in the previous figure. This makes sense for both embedded/immersed submanifolds.

The Tangent Space to a Submanifold

An alternative way to characterise T_pS as a subspace of T_pM is as follows.

Proposition 5.35

Let M be a smooth manifold with immersed (or embedded) submanifold S, and let $p \in S$. A vector $v \in T_pM$ is in T_pS if and only if there is a smooth curve $\gamma\colon J \to M$ whose image is contained in S, and which is also smooth as a map into S, such that $0 \in J$, $\gamma(0) = p$ and $\gamma'(0) = v$.

The proof is an easy exercise.

The Tangent Space to a Submanifold

The following proposition gives us a way to characterise the tangent space in the embedded case.

Proposition 5.37

Let S be an embedded submanifold of a smooth manifold M, and let $p \in S$. As a subspace of T_pM, the tangent space T_pS is characterised by

$$T_pS = \{v \in T_pM : vf = 0 \text{ whenever } f \in C^\infty(M) \text{ and } f \restriction S = 0\}.$$

The Tangent Space to a Submanifold

Proof of 5.37.

Suppose $v \in T_pS \subseteq T_pM$, i.e., $v = d\iota_p(w)$ for some $w \in T_pS$ where $\iota\colon S \to M$ is inclusion. If f is any smooth real-valued function on M that vanishes on S, then $f \circ \iota = 0$, so

$$vf = f\iota_p(w)f = w(f \circ \iota) = 0.$$

Conversely, if $v \in T_pM$ satisfies $vf = 0$ whenever f vanishes on S, we need to prove the existence of a vector $w \in T_pS$ such that $v = d\iota_p(w)$. Let $(x_1, \ldots, x_n)$ be slice coordinates for S in some neighbourhood U of p, so that $U \cap S$ is a subset of U with $x^{k+1} = \cdots = x^n = 0$ and $(x^1, \ldots, x^k)$ are coordinates for $U \cap S$.

Proof of 5.37 (contd.)

Because the inclusion $\iota\colon S \cap U \hookrightarrow M$ has coordinate representation

$$\iota(x^1, \ldots, x^k) = (x^1, \ldots, x^k, 0, \ldots, 0)$$

in these coordinates, it follows that T_pS (or rather, $d\iota_p(T_pS)$) is precisely the subspace of T_pM spanned by $\frac{\partial}{\partial x^1}\Big|_p, \ldots, \frac{\partial}{\partial x^k}\Big|_p$. Writing the coordinate representation of v as

$$v = \sum_{i=1}^{n} v^i \frac{\partial}{\partial x^i}\bigg|_p,$$

we see that $v \in T_pS$ iff $v^i = 0$ for $i > k$.

Proof of 5.37 (contd.)

Let φ be a smooth bump function supported in U that is equal to 1 in a neighbourhood of p. Pick an index $j > k$, and consider the function $f(x) = \varphi(x)x^j$, extended to zero on $M \setminus \operatorname{supp}(\varphi)$. Then f vanishes identically on S, so

$$0 = vf = \sum_{i=1}^{n} v^i \frac{\partial(\varphi(x)x^j)}{\partial x^i}(p) = v^j,$$

as required. $\square$

Thank you!

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Stuff I skipped

- Regular level set theorem and some nice examples on level sets (surface of revolution)
- Some important results on immersed submanifolds
- Restricting maps to submanifolds
- Some nice examples of tangent spaces to submanifolds, cause they required a few more propositions to cover fully
- Submanifolds with boundary

References

- Lee, J. M. *Introduction to Smooth Manifolds*, second ed. Spriger–Verlag, 2013.