Polar Coordinates

Pure Mathematics A-Level

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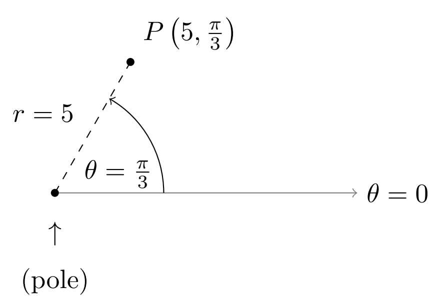
Introduction

In the Cartesian coordinate system with which we are most familiar, we usually specify points in the plane by writing something like $P = (x, y)$, which simply means that the point P can be located by moving x units horizontally and y units vertically from some origin O.

Here we introduce an alternative convention to describe points, where we shall write $P = (r, \theta)$ to specify the *polar* coordinates of a point in the plane, where we call r the *radius* and θ the *angle* of the point. These tell us that to locate P, we start from the *pole* (alternative term for origin), make an angle of θ with the *initial line* (equivalent to the positive x-axis) and move r units in the direction of this angle θ.

For example, the point $P = (5, \frac{\pi}{3})$ is the following:



As a matter of convention and to reduce any confusion, we take $r \geq 0$ and $\theta \in (-\pi, \pi]$. This way, each point has only one representation.[1]

[1] In some cases you may come across negative values of r in your working. When this happens, interpret it as moving $|r|$ units in the *opposite direction* specified by the angle θ.

Common Curves in Polar Coordinates

Here is a list of common curves which one comes across when working with polar coordinates.

- **Circle**

  A circle centred at the pole with radius α is given by the equation $\boxed{r = \alpha}$.

- **Part-line**

  A part-line is a line which is infinite in only one direction. A part-line starting from the pole in the direction specified by the angle θ_0 is given by $\boxed{\theta = \theta_0}$.

- **Horizontal Line**

  A horizontal line k units above the pole (or $|k|$ units below if $k < 0$) is given by the equation $\boxed{r = k \csc \theta}$.

- **Vertical Line**

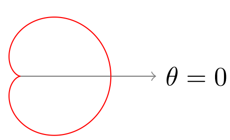
  A vertical line k units to the left of the pole (or $|k|$ units to the right if $k > 0$) is given by the equation $\boxed{r = k \sec \theta}$.

- **Limaçon Curves**

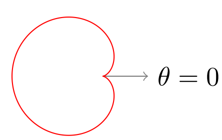
  The limaçon family of curves have equations of the form $\boxed{r = a + b \cos \theta}$ (horizontal) or $\boxed{r = a + b \sin \theta}$ (vertical). Different types of curves arise, depending on the values of a and b.

  - **Cardioid**

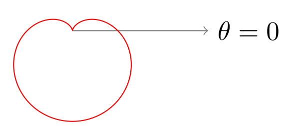
    The cardioid curve (or cusped limaçon) has $|a| = |b|$. Some examples:



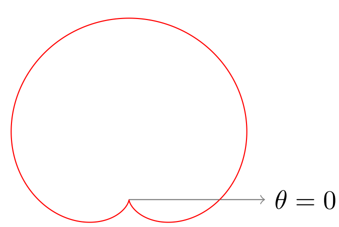
$r = 1 + \cos \theta$



$r = 1 - \cos \theta$



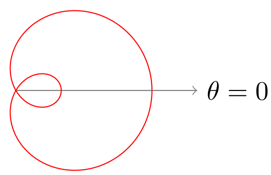
$r = 1 - \sin \theta$



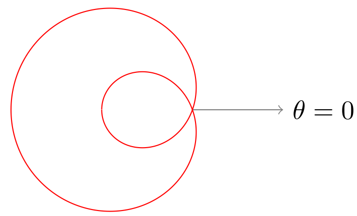
$r = 2 + 2 \sin \theta$

– **Trisectrix**

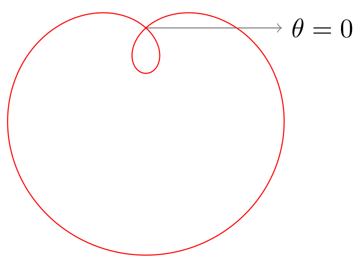
The trisectrix (or looped limaçon) has $|a| < |b|$. In contrast to the Cardioid, the trisectrix has an inner loop. The larger $|b|$ is compared to $|a|$, the larger the inner loop. Some examples:



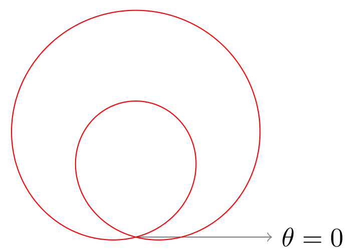
$r = 1 + 2\cos\theta$



$r = 1 - 3\cos\theta$



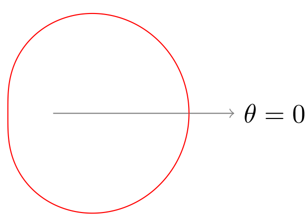
$r = 2 - 3\sin\theta$



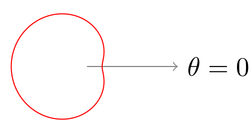
$r = 1 + 4\sin\theta$

– **Dimpled Limaçon**

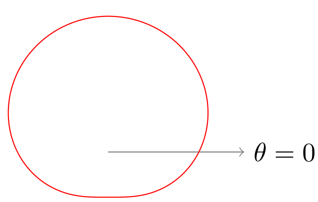
The dimpled limaçon has $|b| < |a| \leq 2|b|$. Some examples:



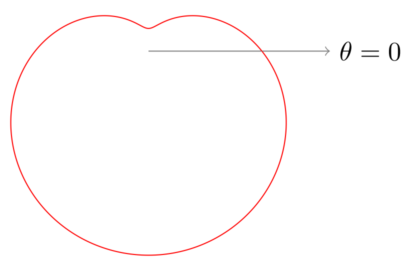
$r = 2 + \cos\theta$



$r = 1 - \frac{2}{3}\cos\theta$



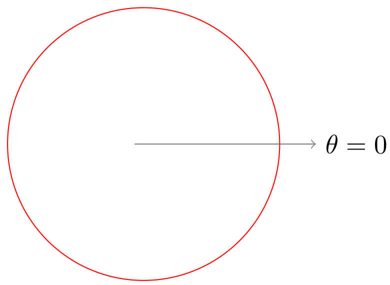
$r = 2 + \sin\theta$



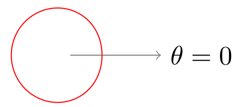
$r = \frac{5}{2} - 2\sin\theta$

– **Convex Limaçon**

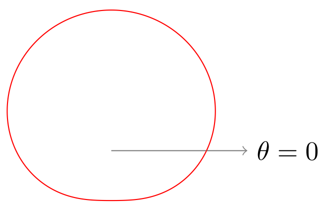
The convex limaçon has $|a| > 2|b|$. Some examples:



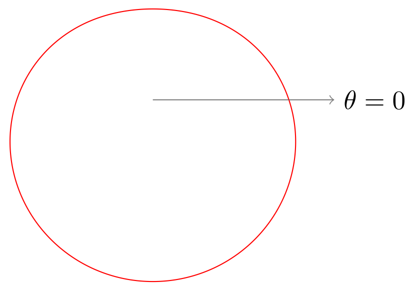
$r = 3 + \frac{1}{5}\cos\theta$



$r = 1 - \frac{1}{3}\cos\theta$



$r = \frac{21}{10} + \sin\theta$



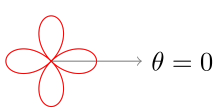
$r = 3 - \sin\theta$

- **Rose**

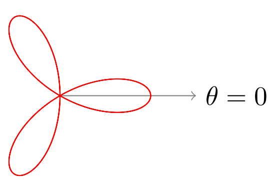
  A rose is a curve with *petals* of equal area. Any equation of the form $\boxed{r = a\cos k\theta}$ or $\boxed{r = a\sin k\theta}$ represents a rose.

  If k is even, then the rose has $2k$ petals. If k is odd, then the rose has k petals. The coefficient a is the *length* of each petal from the pole to its tip. The difference between roses with sin instead of cos is a rotation of $\pi/2k$ radians about the pole.

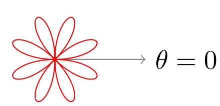
  Some examples:



$r = \cos 2\theta$



$r = 2\cos 3\theta$



$r = \sin 4\theta$

Symmetries of the Sine and Cosine Functions

Suppose a polar curve is given by $r = f(\theta)$ where f depends solely on the value of a sine or cosine of θ. Then we have the following symmetries.

- If $r = f(\cos\theta)$, then the curve is symmetric in the horizontal (x-axis).
- If $r = f(\sin\theta)$, then the curve is symmetric in the vertical (y-axis).
- If $r = f(\cos 2n\theta)$ where $n \in \mathbb{Z}$, then the curve is symmetric in both the vertical and the horizontal.
- If $r = f(\sin 2n\theta)$ where $n \in \mathbb{Z}$, then the curve is symmetric in the diagonal lines $y = \pm x$.

When plotting a polar curve, these symmetries are very useful when considering the range of values for θ. For example, to sketch a curve whose polar equation is dependent solely on $\sin\theta$, we only need consider values of θ in the range $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. Then we can simply reflect the resulting plot in the vertical to obtain the remainder of the curve.

The Relationship Between Polar and Cartesian Coordinates

Suppose a curve $\mathcal{C}$ is given in Cartesian coordinates by $f(x, y) = 0$. By simple trigonometric constructions, one obtains that a point (r, θ) in polar coordinates is equivalent to the point $(r\cos\theta, r\sin\theta)$ in Cartesian coordinates. Therefore the equation

$$f(r\cos\theta, r\sin\theta) = 0$$

equivalently describes the curve $\mathcal{C}$ in polar coordinates.

Conversely, suppose that $\mathcal{C}$ is given in polar coordinates by $f(r, \theta) = 0$. One similarly obtains that the point (x, y) in Cartesian coordinates has polar representation $(\sqrt{x^2 + y^2}, \tan^{-1}\frac{y}{x})$, and therefore the equation

$$f(\sqrt{x^2 + y^2}, \tan^{-1}\tfrac{y}{x}) = 0$$

equivalently describes the curve $\mathcal{C}$ in Cartesian coordinates.

The Area Bounded by a Polar Curve

Suppose a polar curve is given by the equation $r = f(\theta)$. Then the area bounded by the curve and the part lines $\theta = \alpha$ and $\theta = \beta$ is given by the integral

$$A = \frac{1}{2}\int_{\alpha}^{\beta} r^2 \, \mathrm{d}\theta.$$

Tangents to a Polar Curve

Let $r = f(\theta)$ represent a polar curve. We consider only tangents at the pole, vertical and horizontal tangents.

- **Tangents at the Pole**

  Any part-lines tangential to the curve at the pole can be found by solving the equation $\boxed{f(\theta) = 0}$ for values of θ in the range $-\pi < \theta \leq \pi$.

- **Horizontal Tangents**

  Any horizontal tangents to the curve occur when

  $$\frac{\mathrm{d}r}{\mathrm{d}\theta} = -r \cot\theta.$$

  Substituting $f(\theta)$ for r in this equation and solving for θ in the range $-\pi < \theta \leq \pi$, we obtain the angles at which horizontal tangents occur. One can then find the corresponding r-coordinate of each point by substituting the angles in $r = f(\theta)$. Finally for each of these points (r, θ), substitution into the equation $r = k \csc\theta$ of a horizontal line, one can determine the value of k so that this equation represents the tangent at that point.

- **Vertical Tangents**

  Any vertical tangents to the curve occur when

  $$\frac{\mathrm{d}r}{\mathrm{d}\theta} = r \tan\theta.$$

  Substituting $f(\theta)$ for r in this equation and solving for θ in the range $-\pi < \theta \leq \pi$, we obtain the angles at which vertical tangents occur. One can then find the corresponding r-coordinate of each point by substituting the angles in $r = f(\theta)$. Finally for each of these points (r, θ), substitution into the equation $r = k \sec\theta$ of a vertical line, one can determine the value of k so that this equation represents the tangent at that point.