MATHEMATICS TUTORIALS
ĦAL TARXIEN

A Level

20th April 2017 **3 hours**

Pure Mathematics
Paper II

20th April 2017

Solutions

If any errors are found in these solutions, please contact the author by email on `luke@maths.com.mt`, or call 79 000 126.

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Solutions

1. (a) (i) $\dfrac{\mathrm{d}}{\mathrm{d}x}[xy\cos x] = 1 \cdot y \cdot \cos x + x \cdot \dfrac{\mathrm{d}y}{\mathrm{d}x} \cdot \cos x + x \cdot y \cdot -\sin x$

$$= \boxed{x\cos x\frac{\mathrm{d}y}{\mathrm{d}x} + y\cos x - xy\sin x}, \text{ by the product rule.}$$

(ii) $x\dfrac{\mathrm{d}y}{\mathrm{d}x} + y - xy\tan x = x^2$

$$\implies \frac{\mathrm{d}y}{\mathrm{d}x} + \frac{y}{x} - y\tan x = x$$

$$\implies \frac{\mathrm{d}y}{\mathrm{d}x} + \left(\frac{1}{x} - \tan x\right) y = x \qquad (*)$$

Consider
$$\begin{aligned} I(x) &= \exp\int\left(\frac{1}{x} - \tan x\right)\mathrm{d}x \\ &= \exp\left[\int \frac{1}{x}\,\mathrm{d}x - \int \tan x\,\mathrm{d}x\right] \\ &= \exp\left[\ln x - \ln(\sec x)\right] \\ &= \exp\left[\ln\left(\frac{x}{\sec x}\right)\right] \\ &= \exp\left[\ln(x\cos x)\right] = x\cos x \end{aligned}$$

Multiplying the differential equation $(*)$ throughout by $I(x)$:

$$\begin{aligned} &\implies x\cos x\frac{\mathrm{d}y}{\mathrm{d}x} + x\cos x\left(\frac{1}{x} - \tan x\right)y = x^2\cos x \\ &\implies x\cos x\frac{\mathrm{d}y}{\mathrm{d}x} + y\cos x - xy\sin x = x^2\cos x \\ &\implies \frac{\mathrm{d}}{\mathrm{d}x}[xy\cos x] = x^2\cos x \qquad \text{by part (i)} \\ &\implies xy\cos x = \int x^2\cos x\,\mathrm{d}x \end{aligned}$$

$$= uv - \int v\,\mathrm{d}u \qquad \begin{matrix} u = x^2 & \mathrm{d}v = \cos x\,\mathrm{d}x \\ \Rightarrow \mathrm{d}u = 2x\,\mathrm{d}x & \Rightarrow v = \sin x \end{matrix}$$

$$= x^2\sin x - 2\int x\sin x\,\mathrm{d}x$$

$$= x^2\sin x - 2\left(wz - \int z\,\mathrm{d}w\right) \qquad \begin{matrix} w = x & \mathrm{d}z = \sin x\,\mathrm{d}x \\ \Rightarrow \mathrm{d}w = \mathrm{d}x & \Rightarrow z = -\cos x \end{matrix}$$

$$= x^2\sin x - 2\left(-x\cos x + \int \cos x\,\mathrm{d}x\right)$$

$$= x^2\sin x - 2(-x\cos x + \sin x) + c$$

$$\therefore xy\cos x = x^2\sin x + 2x\cos x - 2\sin x + c \qquad \text{(General Solution)}$$

$$\left.\begin{matrix} x = \pi \\ y = 0 \end{matrix}\right\} \implies 0 = \pi^2\sin\pi + 2\pi\cos\pi - 2\sin\pi + c$$

$$\implies c = 2\pi$$

Therefore the *particular solution* is $\boxed{xy\cos x = x^2\sin x + 2x\cos x - 2\sin x + 2\pi}$.

(b) $9\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} - 12\dfrac{\mathrm{d}y}{\mathrm{d}x} + 4y = 8x - 5e^{-x}$

Consider the homogeneous equation $9\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} - 12\dfrac{\mathrm{d}y}{\mathrm{d}x} + 4y = 0$. Then $y = e^{kx}$ is a solution.

$$\begin{aligned}\text{Auxiliary Equation: } & 9k^2 - 12k + 4 = 0\\ \implies & (3k-2)^2 = 0\\ \implies & k = \frac{2}{3} \quad \text{(twice)}.\end{aligned}$$

Therefore we take $\mathrm{cf}(x) = e^{2/3x}(c_1 + c_2x)$ as the complementary function.

We want a trial solution $\mathrm{ts}(x)$ whose linear combination with its first and second order derivatives results in $8x - 5e^{-x}$. Thus we take $\mathrm{ts}(x) = \lambda x + \mu + \eta e^{-x}$, so:

$$\begin{aligned}&\implies \mathrm{ts}'(x) = \lambda - \eta e^{-x}\\ &\implies \mathrm{ts}''(x) = \eta e^{-x}\end{aligned}$$

Now we try substitute these in the equation:

$$\begin{aligned}&9\,\mathrm{ts}''(x) - 12\,\mathrm{ts}'(x) + 4\,\mathrm{ts}(x) = 8x - 5e^{-x}\\ \implies& 9\eta e^{-x} - 12(\lambda - \eta e^{-x}) + 4(\lambda x + \mu + \eta e^{-x}) = 8x - 5e^{-x}\\ \implies& 9\eta e^{-x} - 12\lambda + 12\eta e^{-x} + 4\lambda x + 4\mu + 4\eta e^{-x} = 8x - 5e^{-x}\\ \implies& 4\lambda x + (4\mu - 12\lambda) + 25\eta e^{-x} = 8x - 5e^{-x}\end{aligned}$$

Comparing coefficients of $x \implies 4\lambda = 8 \implies \lambda = 2$.
Comparing constant terms $\implies 4\mu - 12\lambda = 0 \implies 4\mu - 24 = 0 \implies \mu = 6$.
Comparing coefficients of $e^{-x} \implies 25\eta = -5 \implies \eta = -1/5$.

Therefore our trial solution worked, and we have obtained the particular integral $\mathrm{pi}(x) = 2x + 6 - e^{-x}/5$. The general solution is given by $y = \mathrm{cf}(x) + \mathrm{pi}(x)$, i.e.

$$y = e^{2/3x}(c_1 + c_2x) + 2x + 6 - \frac{e^{-x}}{5}.$$

$$\left.\begin{aligned}x &= 0\\ y &= \tfrac{14}{5}\end{aligned}\right\} \implies \frac{14}{5} = e^0(c_1 + 0) + 0 + 6 - \frac{e^0}{5}$$
$$\implies c_1 = -3.$$

Now if we differentiate the general solution, we obtain

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2}{3}e^{2x/3}(c_1 + c_2x) + c_2e^{2x/3} + 2 + \frac{e^{-x}}{5}$$
$$\left.\begin{aligned}x &= 0\\ \tfrac{\mathrm{d}y}{\mathrm{d}x} &= \tfrac{11}{5}\end{aligned}\right\} \implies \frac{11}{5} = \frac{2}{3}e^0(-3+0) + c_2e^0 + 2 + \frac{e^0}{5}$$
$$\implies c_2 = 2.$$

Therefore the *particular solution* is $\boxed{y = e^{2/3x}(2x - 3) + 2x + 6 - \dfrac{e^{-x}}{5}}$.

[7, 8 marks]

2. (a) To show: $I_n = n(n-1)I_{n-2} - x^{n-1}e^{-x}(x+n)$, where $I_n = \Gamma_n(x)$.

$$\begin{aligned}
I_n &= \int_0^x t^n e^{-t}\,\mathrm{d}t \\
&= uv\big|_0^x - \int_0^x v\,\mathrm{d}u && \begin{aligned} u &= t^n & \mathrm{d}v &= e^{-t}\,\mathrm{d}t \\ \Rightarrow \mathrm{d}u &= nt^{n-1}\,\mathrm{d}t & v &= -e^{-t} \end{aligned} \\
&= -t^n e^{-t}\big|_0^x + n\int_0^x t^{n-1}e^{-t}\,\mathrm{d}t \\
&= -x^n e^{-x} + n\left(wz\big|_0^x - \int_0^x z\,\mathrm{d}w\right) && \begin{aligned} w &= t^{n-1} & \mathrm{d}z &= e^{-t}\,\mathrm{d}t \\ \Rightarrow \mathrm{d}w &= (n-1)t^{n-2}\,\mathrm{d}t & z &= -e^{-t} \end{aligned} \\
&= -x^n e^{-x} + n\left(-t^{n-1}e^{-t}\big|_0^x + (n-1)\int_0^x t^{n-2}e^{-t}\,\mathrm{d}t\right) \\
&= -x^n e^{-x} + n(-x^{n-1}e^{-x} + (n-1)I_{n-2}) \\
&= -x^n e^{-x} - nx^{n-1}e^{-x} + n(n-1)I_{n-2} \\
&= -x^{n-1}e^{-x}(x+n) + n(n-1)I_{n-2},
\end{aligned}$$

as required. □

(b) $\Gamma_7(1) = I_7$ with $x = 1$. By the reduction formula in part (a),

$$I_7 = 7(6)I_5 - e^{-1}(1+7) = 42I_5 - \frac{8}{e} \tag{1}$$

$$I_5 = 5(4)I_3 - e^{-1}(1+5) = 20I_3 - \frac{6}{e} \tag{2}$$

$$I_3 = 3(2)I_1 - e^{-1}(1+3) = 6I_1 - \frac{4}{e} \tag{3}$$

$$\begin{aligned}
\text{But } I_1 &= \int_0^1 t\,e^{-t}\,\mathrm{d}t = uv\big|_0^1 - \int_0^1 v\,\mathrm{d}u && \begin{aligned} u &= t & \mathrm{d}v &= e^{-t}\,\mathrm{d}t \\ \Rightarrow \mathrm{d}u &= \mathrm{d}t & v &= -e^{-t} \end{aligned} \\
&= -te^{-t}\big|_0^1 + \int e^{-t}\,\mathrm{d}t \\
&= -e^{-1} - e^{-t}\big|_0^1 = -e^{-1} - e^{-1} + 1 = 1 - 2e^{-1}
\end{aligned}$$

Substituting $I_1 = 1 - 2e^{-1}$ into (3) $\implies I_3 = 6 - 16/e$.
Substituting $I_3 = 6 - 16/e$ into (2) $\implies I_5 = 120 - 326/e$.
Substituting $I_5 = 120 - 326/e$ into (1) $\implies I_7 = 5040 - 13\,700/e = \boxed{7! - 13\,700/e}$. □

(c) The volume of the solid when rotated through $360°$ is given by the integral $\pi\int_a^b y^2\,\mathrm{d}x$. So the desired volume is half of that, i.e.

$$\begin{aligned}
V &= \frac{\pi}{2}\int_0^2 y^2\,\mathrm{d}x = \frac{\pi}{2}\int_0^2 (x^{5/2}e^{-x/2})^2\,\mathrm{d}x \\
&= \frac{\pi}{2}\int_0^2 x^5 e^{-x}\,\mathrm{d}x = \frac{\pi}{2}\Gamma_5(2)
\end{aligned}$$

Similar to part (b), this can be found with two iterations of the formula from part (a) to arrive at result of $\boxed{60\pi - 463\pi/e^2}$.

[5, 4, 6 marks]

3. (a) Consider the equation $y = f(x)$:

$$\begin{aligned}
&\implies y = \frac{6x - x^2}{x^2 - 6x + 5} \\
&\implies y(x^2 - 6x + 5) = 6x - x^2 \\
&\implies yx^2 - 6yx + 5y = 6x - x^2 \\
&\implies (y+1)x^2 - 6(y+1)x + 5y = 0 \qquad (*)
\end{aligned}$$

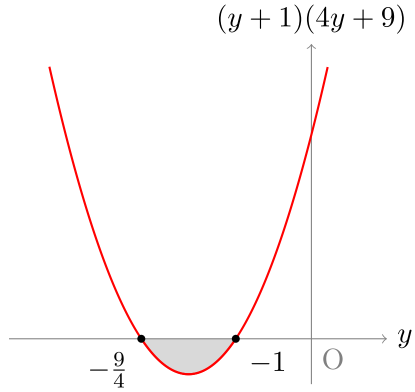
This equivalent equation of our curve is a quadratic in x. If we determine the range of values of y for which this quadratic equation has no roots, we will be finding the range of values of y for which no value of x gives $f(x) = y$; i.e. where the curve does not exist.

Therefore no part of the curve exists when the quadratic discriminant $\Delta < 0$:

$$\begin{aligned}
&\implies 36(y+1)^2 - 20y(y+1) < 0 \\
&\implies 36(y^2 + 2y + 1) - 20y^2 - 20y < 0 \\
&\implies 16y^2 + 52y + 36 < 0 \\
&\implies 4y^2 + 13y + 9 < 0 \\
&\implies (y+1)(4y+9) < 0
\end{aligned}$$

From the graph,

$$\therefore \boxed{-\frac{9}{4} < y < -1}$$



(b) Since the curve exists everywhere else other than the region $-\frac{9}{4} < y < -1$, then any extrema must occur at one of the points where $y = -\frac{9}{4}$ and $y = -1$. (We can safely say this only because our rational function is quadratic).

Thus, we substitute $y = -1$ and $y = -\frac{4}{9}$ in the equation $(*)$. Immediately we realise that substituting $y = -1$ leads to a contradictory result, so there are no extrema with y-coordinate -1. Substituting $y = -\frac{9}{4}$ however yields the equation

$$\begin{aligned}
&-\frac{5}{4}x^2 - 6\left(-\frac{5}{4}\right)x + 5\left(-\frac{9}{4}\right) = 0 \\
&\implies 5x^2 - 30x + 45 = 0 \\
&\implies x^2 - 6x + 9 = 0 \\
&\implies (x-3)^2 = 0 \\
&\implies x = 3 \quad \text{(twice)}
\end{aligned}$$

Therefore a turning point occurs at $\boxed{\left(3, -\frac{9}{4}\right)}$.

Now, to determine asymptotes. Vertical asymptotes occur when the denominator is zero:

$$\begin{aligned}
&x^2 - 6x + 5 = 0 \\
&\implies (x-1)(x-5) = 0 \\
&\implies x = 1 \quad \text{or} \quad x = 5
\end{aligned}$$

Therefore the equations of the vertical asymptotes are $\boxed{x = 1 \quad \text{and} \quad x = 5}$. Horizontal or oblique asymptotes occur as $x \to \pm\infty$. We therefore consider the limits of f as $x \to +\infty$ and as $x \to -\infty$:

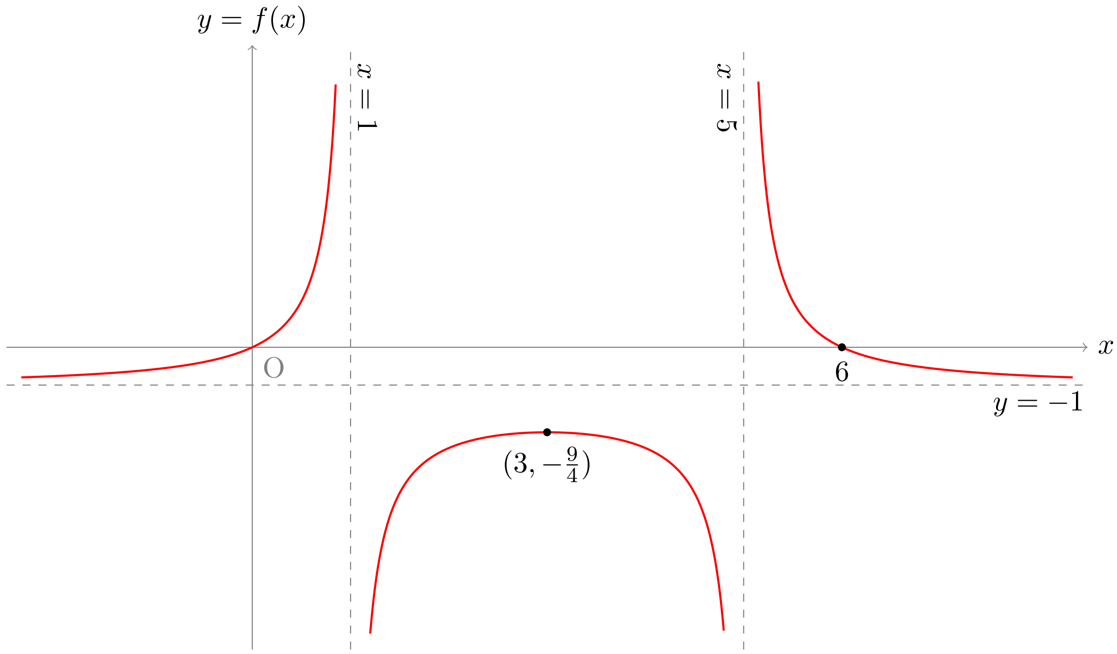
$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{6x - x^2}{x^2 - 6x + 5} = \lim_{x\to\infty} \left(-1 + \frac{5}{x^2 - 6x + 5}\right) = -1,$$

Considering the limit as $x \to -\infty$ gives the same result (This is always the case for such functions). Therefore $\boxed{y = -1}$ is the only horizontal asymptote to the curve.

(c) For the y-intercept, set $x = 0 \implies f(0) = 0$.

For the x-intercepts, we solve $f(x) = 0 \implies 6x - x^2 = 0 \implies x(6 - x) = 0 \implies x = 0$ or $x = 6$.

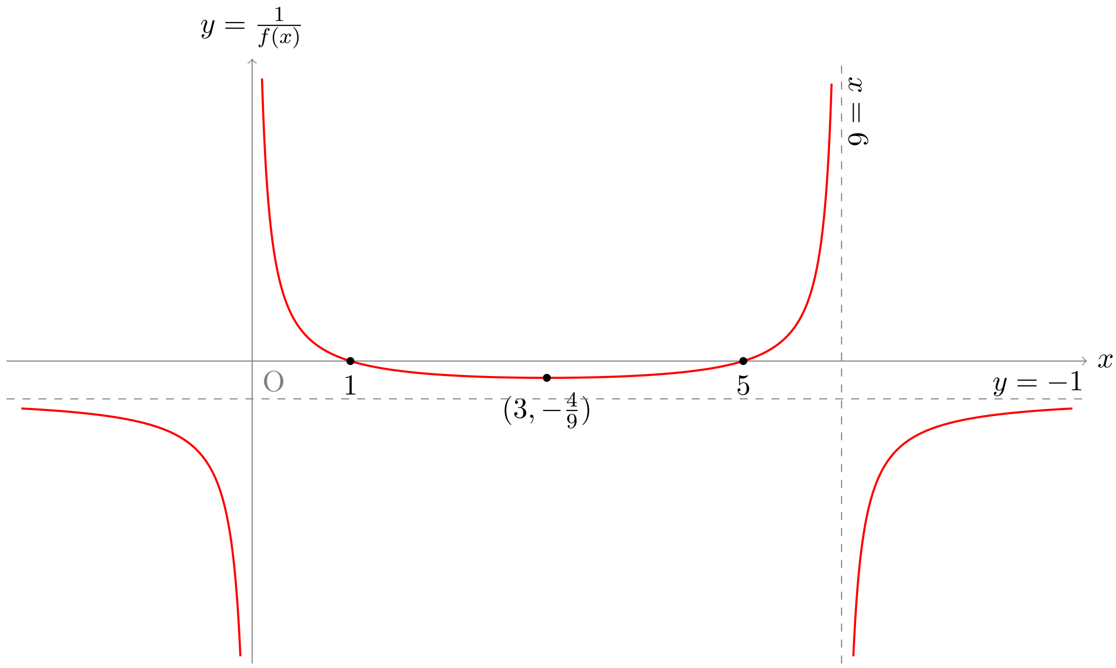
Sketch:



(d) For the graph $y = \dfrac{1}{f(x)}$, we make the following considerations.

- Any x-intercepts of the curve $y = f(x)$ are roots of $y = 1/f(x)$ and vice-versa. (So now, $x = 0$ and $x = 6$ are asymptotes, whereas $x = 1$ and $x = 5$ are roots).
- Each respective part of the curve $y = f(x)$ remains in the same quadrant when considering the curve $y = 1/f(x)$.
- If $f(x) \to \infty$, then $1/f(x) \to 0^+$ (from above) and vice-versa. Similarly, if $f(x) \to -\infty$, then $1/f(x) \to 0^-$ (from below) and vice-versa.
- If $y = f(x)$ has a maximum turning point at (x_0, y_0), then $y = 1/f(x)$ has a minimum turning point at $\left(x_0, \frac{1}{y_0}\right)$ and vice-versa. (So now, a maximum turning point occurs at $(3, -\frac{4}{9})$).
- Any horizontal asymptotes given by $y = a$ are still present in the curve $y = 1/f(x)$, however they are shifted to $y = \frac{1}{a}$. (The asymptote at $y = -1$ remains at $y = -1$).

With these in mind, we can proceed to sketch $y = 1/f(x)$.



[4, 4, 4, 3 marks]

4. (a) First, we show that $1 + 2 + \cdots + n = n(n+1)/2$.

For the base case, take $n = 1$. Clearly $1 = \dfrac{1(1+1)}{2}$, as required.

Now suppose the statement holds for some $n = k \in \mathbb{N}$, i.e. that $1+2+\cdots+k = k(k+1)/2$.

$$\begin{aligned} 1 + 2 + \cdots + k + (k+1) &= \frac{k(k+1)}{2} + (k+1) \qquad \text{(by the hypothesis)} \\ &= \frac{k(k+1) + 2(k+1)}{2} \\ &= \frac{(k+1)(k+2)}{2} \\ &= \frac{(k+1)((k+1)+1)}{2}, \end{aligned}$$

as required. □

Now we proceed to show that $(1 + 2 + 3 + \cdots + n)^2 = 1^3 + 2^3 + 3^3 + \cdots + n^3$. Again, for the base case, we take $n = 1$. Clearly $(1)^2 = 1 = 1^3$, as required.

Now suppose the statement holds for some $n = k \in \mathbb{N}$.

$$\begin{aligned} (1 + 2 + \cdots + k + (k+1))^2 &= (1 + 2 + \cdots + k)^2 + 2(1 + 2 + \cdots + k)(k+1) + (k+1)^2 \\ &= 1^3 + 2^3 + \cdots + k^3 + (k+1)\left[2(1 + 2 + \cdots + k) + (k+1)\right] \\ &\qquad\qquad \text{(by the hypothesis)} \\ &= 1^2 + 2^3 + \cdots + k^3 + (k+1)(k(k+1) + (k+1)) \\ &\qquad\qquad \text{(by the previous result)} \\ &= 1^2 + 2^3 + \cdots + k^3 + (k+1)(k^2 + 2k + 1) \\ &= 1^2 + 2^3 + \cdots + k^3 + (k+1)(k+1)^2 \\ &= 1^2 + 2^3 + \cdots + k^3 + (k+1)^3, \end{aligned}$$

as required. □

(b) To show: $\displaystyle\sum_{r=1}^{2^n} \frac{1}{r} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^n - 2} + \frac{1}{2^n - 1} + \frac{1}{2^n} \geq 1 + \frac{n}{2}$.

For the base case, take $n = 0$. We have $\displaystyle\sum_{r=1}^{2^0} \frac{1}{r} = 1 \geq 1 + \frac{0}{2}$, as required.

Now suppose the statement holds for some $n = k \in \mathbb{N}$.

$$\begin{aligned}
\sum_{r=1}^{2^{k+1}} \frac{1}{r} &= \sum_{r=1}^{2^k} \frac{1}{r} + \sum_{r=2^k+1}^{2^{k+1}} \frac{1}{r} \\
&\geq 1 + \frac{k}{2} + \sum_{r=2^k+1}^{2^{k+1}} \frac{1}{r} \qquad \text{(by the hypothesis)} \\
&= 1 + \frac{k}{2} + \frac{1}{2^k+1} + \frac{1}{2^k+2} + \cdots + \frac{1}{2^{k+1}} \\
&\geq 1 + \frac{k}{2} + \frac{1}{2^{k+1}} + \frac{1}{2^{k+1}} + \cdots + \frac{1}{2^{k+1}} \\
&= 1 + \frac{k}{2} + 2^k \left(\frac{1}{2^{k+1}} \right) \qquad \text{(since there are } 2^k \text{ terms)} \\
&= 1 + \frac{k}{2} + \frac{2^k}{2^{k+1}} \\
&= 1 + \frac{k}{2} + \frac{1}{2} \\
&= 1 + \frac{k+1}{2},
\end{aligned}$$

as required. □

Now the harmonic series $\displaystyle\sum_{r=1}^{\infty} \frac{1}{r} = \lim_{n \to \infty} \sum_{k=1}^{2^n} \frac{1}{r} \geq \lim_{n \to \infty} \left(1 + \frac{n}{2}\right) \to \infty$, by the result above. Hence the harmonic series diverges.

[8, 7 marks]

5. (a) $\sin x + \cos x = x \implies \sin x + \cos x - x = 0$. Define $f(x) := \sin x + \cos x - x$. Then $f(1) \approx 0.3818$, and $f(2) \approx -1.5069$. Since f is clearly continuous in the interval $(1, 2)$ and a change in sign occurs between $f(1)$ and $f(2)$, then at some point in $1 < x < 2$, we must have $f(x) = 0$, i.e. $\sin x + \cos x - x = 0$, i.e. $\sin x + \cos x = x$. □

Now $f'(x) = \cos x - \sin x - 1$. Applying the Newton-Raphson method twice:

$$\begin{aligned}
x_1 &= x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{0.3818}{-1.3012} = 1.2934 \\
x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} = 1.2934 - \frac{-0.0578}{-1.6879} = 1.2592
\end{aligned}$$

Therefore a better approximation for the solution is $\boxed{x_2 = 1.2592}$.

(b) (i) $\displaystyle\mathrm{erf}(0.6) = \frac{2}{\sqrt{\pi}} \int_0^{0.6} e^{-t^2}\, \mathrm{d}t$.

t	0	0.1	0.2	0.3	0.4	0.5	0.6
e^{-t^2}	1	0.9901	0.9608	0.9139	0.8521	0.7788	0.6977

Applying Simpson's rule:

$$\begin{aligned}\int_0^{0.6} e^{-t^2}\,\mathrm{d}t &\approx \frac{0.1}{3}\left[1+0.6977+4(0.9901+0.9139+0.7788)+2(0.9608+0.8521)\right]\\ &= 0.535157\end{aligned}$$

Thus $\operatorname{erf}(0.6) \approx \frac{2}{\sqrt{\pi}}(0.535157) = \boxed{0.60386}$.

(ii) Using the standard Maclaurin expansion of e^x, we have

$$\begin{aligned} e^{-t^2} &= 1+(-t^2)+\frac{(-t^2)^2}{2!}+\frac{(-t^2)^3}{3!}+\cdots\\ &= 1-t^2+\frac{t^4}{2}-\frac{t^6}{6}+\mathcal{O}(t^8)\\ \implies \frac{2}{\sqrt{\pi}}e^{-t^2} &= \frac{1}{3\sqrt{\pi}}\left(6-6t^2+3t^4-t^6\right)+\mathcal{O}(x^8).\\ \text{Thus }\operatorname{erf}(0.6) &= \frac{2}{\sqrt{\pi}}\int_0^{0.6} e^{-t^2}\,\mathrm{d}t\\ &\approx \frac{1}{3\sqrt{\pi}}\int_0^{0.6}(6-6t^2+3t^4-t^6)\,\mathrm{d}t\\ &= \frac{1}{3\sqrt{\pi}}\left(6t-2t^3+\frac{3t^5}{5}-\frac{t^7}{7}\right)\Bigg|_0^{0.6}\\ &= \boxed{0.60381}\end{aligned}$$

(iii) $\%\text{ Error} = \dfrac{|\text{Actual value} - \text{Value obtained}|}{\text{Actual value}} \times 100\%.$

Thus for Simpson's rule, the %-error is practically 0%, and using the series expansion, the % error is 0.008%. Using Simpson's rule gave a more accurate result, however both are very accurate.

[7, 8 marks]

6. (a) First, we substitute $\mathbf{A}$ in the given matrix equation.

$$\begin{aligned}&(\mathbf{A}-\mathbf{I})(\mathbf{A}+2\mathbf{I}) = \mathbf{0}_{3\times3}\\ \implies &\left[\begin{pmatrix}-3&12&4\\-2&7&2\\5&a&b\end{pmatrix}-\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\right]\left[\begin{pmatrix}-3&12&4\\-2&7&2\\5&a&b\end{pmatrix}+\begin{pmatrix}2&0&0\\0&2&0\\0&0&2\end{pmatrix}\right] = \mathbf{0}_{3\times3}\\ \implies &\begin{pmatrix}-4&12&4\\-2&6&2\\5&a&b-1\end{pmatrix}\begin{pmatrix}-1&12&4\\-2&9&2\\5&a&b+2\end{pmatrix} = \mathbf{0}_{3\times3}\\ \implies &\begin{pmatrix}0&4a+60&4(b+2)+8\\0&2a+30&2(b+2)+4\\-2a+5(b-1)-5&(b-1)a+9a+60&2a+(b-1)(b+2)+20\end{pmatrix} = \mathbf{0}_{3\times3}\end{aligned}$$

Comparing entries with the zero matrix, we can quickly see that we have $\boxed{a=-15 \text{ and } b=-4}$.

Now the given equation expands to $\mathbf{A}^2+\mathbf{A}-2\mathbf{I} = \mathbf{0}_{3\times3}$. Premultiplying both sides by $\mathbf{A}^{-1}$ gives $\mathbf{A}^{-1}(\mathbf{A}^2+\mathbf{A}-2\mathbf{I}) = \mathbf{A}^{-1}\,\mathbf{0}_{3\times3} \implies \mathbf{A}^{-1}\mathbf{A}^2+\mathbf{A}^{-1}\mathbf{A}-2\mathbf{A}^{-1}\mathbf{I} = \mathbf{0}_{3\times3} \implies \mathbf{A}+\mathbf{I}-2\mathbf{A}^{-1} = \mathbf{0}_{3\times3}$. Therefore $\mathbf{A}^{-1} = \frac{1}{2}(\mathbf{A}+\mathbf{I})$.

Thus $\mathbf{A}^{-1} = \frac{1}{2}(\mathbf{A}+\mathbf{I}) = \boxed{\frac{1}{2}\begin{pmatrix} -2 & 12 & 4 \\ -2 & 8 & 2 \\ 5 & -15 & -3 \end{pmatrix}}$.

(b) (i) The trivial solution $\mathbf{x} = (0,0,0)$ is always a valid solution to the system.

Now consider the augmented matrix $\mathbf{B}|\mathbf{0}$:

$$\mathbf{B}|\mathbf{0} = \left(\begin{array}{ccc|c} -3 & 12 & 4 & 0 \\ -2 & 7 & 2 & 0 \\ 5 & a & 6 & 0 \end{array}\right)$$

$2R_1 + (-3)R_2 \to R_2$
$5R_1 + 3R_3 \to R_3$

$$\sim \left(\begin{array}{ccc|c} -3 & 12 & 4 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & 60+3a & 38 & 0 \end{array}\right)$$

$(-19)R_2 + R_3 \to R_3$

$$\sim \left(\begin{array}{ccc|c} -3 & 12 & 4 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & 3+3a & 0 & 0 \end{array}\right)$$

Observe that the third row corresponds to the equation $0x + (3+3a)y + 0z = 0$. If we take $a = -1$, then the equation has infinitely many solutions. If $a \neq -1$, then we must take $y = 0$, and consequently, the other rows will give us $x = z = 0$. Therefore only the value $a = -1$ gives more than one solution. □

(ii) From part (i), $\boxed{a=-1}$. Now from R_2, we get $3y + 2z = 0 \implies z = -\frac{3}{2}y$. Similarly, from R_1, we get $-3x+12y+4z = 0$. Making y subject from the previous equation and substituting gives $-3x + 12(-\frac{2}{3}z) + 4z = 0 \implies -3x - 8z + 4z = 0 \implies z = -\frac{3}{4}x$.

Thus combining these equations:

$$-\frac{3}{4}x = -\frac{3}{2}y = z$$
$$\implies \frac{x-0}{-\frac{4}{3}} = \frac{y-0}{-\frac{2}{3}} = \frac{z-0}{1}$$

This corresponds to the Cartesian equation $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$ of a line in $\mathbb{R}^3$ with initial point (x_1, y_1, z_1) and direction vector (a,b,c). The corresponding vector equation is therefore $\boxed{\mathbf{r} = \lambda(4\mathbf{i} + 2\mathbf{j} - 3\mathbf{k})}$. This line represents the intersection of the three planes $-3x + 12y + 4z = 0$, $-2x + 7y + 2z = 0$, $5x - y + 6z = 0$.

[6, 9 marks]

7. (a) (i) Let us choose 10 out of the 20 available books to place in one crate, and place the remaining books in the other crate. This can be done in $\binom{20}{10}$ ways. However, this counts equivalent distributions: suppose we chose 10 particular books and placed them in the first crate. If we, instead, chose the 10 books which were left over by the first choice, and placed them in the first crate instead, this would be equivalent (since we are not distinguishing between the two crates). Thus for each possible distribution, there is another equivalent distribution simply obtained by switching crates. So the actual number of distributions is $\frac{1}{2}\binom{20}{10} = \boxed{92\,378 \text{ ways}}$.

(ii) Firstly, in how many ways can the books be distributed? Well, each book can either be in the first crate, or in the second crate. Thus there are two possibilities for each book – i.e. we have $2 \times 2 \times \cdots \times 2 = 2^{20}$ possibilities. But as we have said in part (a), we are not distinguishing between crates – so half of these are equivalent. Therefore there are 2^{19} different distributions.

Now, we can choose 6 of the 12 books to place in the first crate in $\binom{12}{6}$ ways. Observe that this number does not include doubly-counted distributions (unlike what we have counted so far), since these are not the only books we are distributing. The first crate will contain other books which the second does not; so placing any 6 RLS books in the first crate corresponds to a different distribution than placing them in the second.

Therefore the probability is $\dfrac{1}{2^{19}}\dbinom{12}{6} = \boxed{\dfrac{231}{131\,072}}$.

(iii) Again, in this scenario, the crates *are* distinct; since they contain books already. Thus placing the first poetry book in the first crate corresponds to a different distribution than if we were to place it in the second crate. Now, for each crate to contain at least one poetry book, then exactly one of them contains two. Suppose the first crate contains two poetry books. Then we can choose the books in $\binom{4}{2}\binom{2}{1}\binom{1}{1}$ ways. If, on the other hand, the second crate contained the two, then we have $\binom{4}{1}\binom{3}{2}\binom{1}{1}$ ways. Finally, if the third crate contains the two, then we have $\binom{4}{1}\binom{3}{1}\binom{2}{2}$ ways. Thus the total number of admissible poetry book distributions is $\binom{4}{2}\binom{2}{1}\binom{1}{1} + \binom{4}{1}\binom{3}{2}\binom{1}{1} + \binom{4}{1}\binom{3}{1}\binom{2}{2} = 36$ ways.

Now the denominator. How many ways can we distribute 4 distinct books into 3 distinct crates? Well, this time think of it from the point of view of the books. Each book chooses a crate, where repetition is allowed. We have 3 crates to choose from, and we are choosing 4 times. We know that with repetition allowed and order important (order is important since we are distinguishing between book 1, book 2, etc), this number is n^k. Thus the denominator is $3^4 = 81$.

Thus the probability is $\dfrac{36}{81} = \boxed{\dfrac{4}{9}}$.

(b) (i) We know that $\mathrm{P}(A \cap B) = \mathrm{P}(B \cap A) \implies \mathrm{P}(A)\,\mathrm{P}(B|A) = \mathrm{P}(B)\,\mathrm{P}(A|B)$. Dividing by $\mathrm{P}(B)$ gives Bayes' theorem.

(ii) Let P denote the event that a student passes a test, and let S denote the event that they studied. By Bayes' theorem: $\mathrm{P}(P|S) = \dfrac{\mathrm{P}(S|P)\,\mathrm{P}(P)}{\mathrm{P}(S)} = \dfrac{96\% \times 45\%}{65\%} = \boxed{66.45\%}$.

[2, 3, 4, 3, 3 marks]

8. (a) We use the results $2\cos n\theta \equiv z^n + \frac{1}{z^n}$ and $2i\sin n\theta \equiv z^n - \frac{1}{z^n}$ where $z = \cos\theta + i\sin\theta$. Consider:

$$\begin{aligned}
(2i\sin\theta)^6 &\equiv \left(z - \frac{1}{z}\right)^6 \\
\implies -64\sin^6\theta &\equiv z^6 - 6z^4 + 15z^2 - 20 + \frac{15}{z^2} - \frac{6}{z^4} + \frac{1}{z^6} \\
&\equiv \left(z^6 + \frac{1}{z^6}\right) - 6\left(z^4 + \frac{1}{z^4}\right) + 15\left(z^2 + \frac{1}{z^2}\right) - 20 \\
&\equiv 2\cos 6\theta - 6(2\cos 4\theta) + 15(2\cos 2\theta) - 20 \\
\implies 32\sin^6\theta &\equiv 10 - 15\cos 2\theta + 6\cos 4\theta - \cos 6\theta.
\end{aligned}$$

We proceed similarly to obtain the identity for $32\cos^6\theta$:

$$\begin{aligned}
(2\cos\theta)^6 &\equiv \left(z+\frac{1}{z}\right)^6\\
\implies 64\cos^6\theta &\equiv z^6+6z^4+15z^2+20+\frac{15}{z^2}+\frac{6}{z^4}+\frac{1}{z^6}\\
&\equiv \left(z^6+\frac{1}{z^6}\right)+6\left(z^4+\frac{1}{z^4}\right)+15\left(z^2+\frac{1}{z^2}\right)+20\\
&\equiv 2\cos 6\theta+-6(2\cos 4\theta)+15(2\cos 2\theta)-20\\
\implies 32\cos^6\theta &\equiv 10+15\cos 2\theta+6\cos 4\theta+\cos 6\theta.
\end{aligned}$$

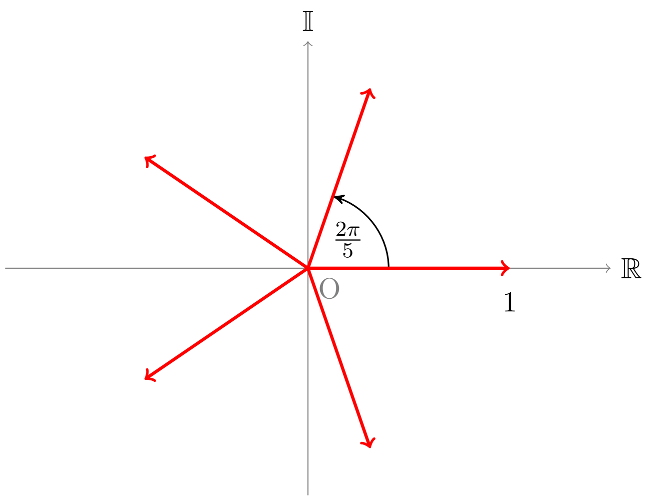
Adding the two identities, we obtain $32\sin^6\theta+32\cos^6\theta\equiv 20+12\cos 4\theta$, which simplifies to $8(\sin^6\theta+\cos^6\theta)\equiv 5+3\cos 4\theta$, as required. Therefore the required integral becomes

$$\begin{aligned}
\int_0^{64\pi}(\sin^6\theta+\cos^6\theta)^2\,\mathrm{d}\theta &= \frac{1}{64}\int_0^{64\pi}(5+3\cos 4\theta)^2\,\mathrm{d}\theta\\
&= \frac{1}{64}\int_0^{64\pi}(25+30\cos 4\theta+9\cos^2 4\theta)\,\mathrm{d}\theta\\
&= \frac{1}{64}\left(25\theta+\frac{15}{2}\sin 4\theta+\frac{9}{2}\int_0^{64\pi}(\cos 8\theta+1)\,\mathrm{d}\theta\right)\Bigg|_0^{64\pi}\\
&= \frac{25\theta}{64}+\frac{15\sin 4\theta}{128}+\frac{9\sin 8\theta}{1024}+\frac{9\theta}{128}\Bigg|_0^{64\pi}\\
&= 25\pi+\frac{9\pi}{2}=\boxed{\frac{59\pi}{2}}.
\end{aligned}$$

(b) The fifth roots of unity are the solutions to the equation $z^5=1$. Clearly, each solution has $|z|=\sqrt[5]{1}=1$. Thus all we have to determine are the different possible values of $\arg z$. We know that the possible values of $\arg z$, where $z^n=r(\cos\alpha+i\sin\alpha)$, are given by

$$\theta=\frac{2k\pi\pm\alpha}{n},\quad k\in\mathbb{Z},$$

so long as $-\pi<\theta\leq\pi$. In our case, we have $\alpha=0$ and $n=1$, so the different possible values are $\theta=\left\{-\frac{4\pi}{5},-\frac{2\pi}{5},0,\frac{2\pi}{5},\frac{4\pi}{5}\right\}$. Therefore the fifth roots of unity are $\boxed{z=\cos\theta+i\sin\theta,\text{ where }\theta=\left\{-\frac{4\pi}{5},-\frac{2\pi}{5},0,\frac{2\pi}{5},\frac{4\pi}{5}\right\}}$.



(i) $z = 1$ is obtained when we take $\theta = 0$. Now let $\omega = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$. Then we get $\omega^2 = (\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5})^2 = \cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}$, by De Moivre's theorem, which is another root. Similarly, $\omega^3 = (\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5})^3 = \cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5}$, again by De Moivre's theorem. Now $\frac{6\pi}{5}$ is out of the range $-\pi < \theta \le \pi$, but we can express it equivalently as $\frac{6\pi}{5} - 2\pi = -\frac{4\pi}{5}$, therefore ω^3 corresponds to the root given by taking $\theta = -\frac{4\pi}{5}$. Finally, $\omega^4 = (\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5})^4 = \cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5}$, whose argument is equivalent to $\frac{8\pi}{5} - 2\pi = \frac{-2\pi}{4}$, the remaining value of θ.

Therefore the roots can be expressed as 1, ω, ω^2, ω^3 and ω^4. $\square$

(ii) $(1-\omega)(1+\omega+\omega^2+\omega^3+\omega^4) = 1-\omega^5 = 0$ by definition of the the fifth roots of unity. Thus either $1-\omega = 0$, i.e. $\omega = 1$, or $1+\omega+\omega^2+\omega^3+\omega^4 = 0$. Taking $\omega = 1$ will not give us a meaningful result in this case; so we may discard the former. Taking ω as in part (i), we get that the sum of all the fifth roots of unity is zero. $\square$

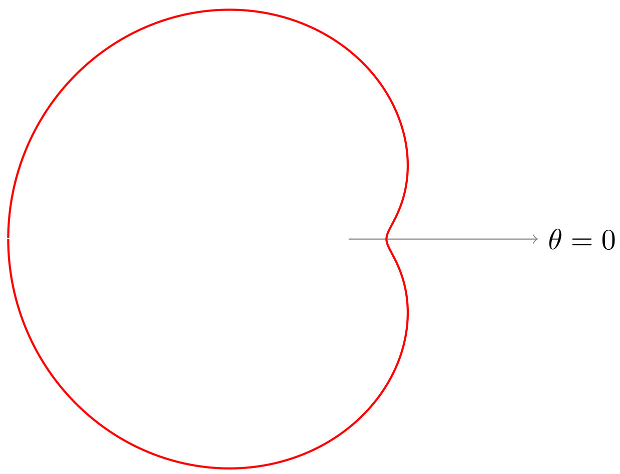
(iii) If we refer to the diagram drawn in part (i), the pentagon is obtained simply by joining the vertices together, giving rise to 5 isosceles triangles, each with apex angle $\frac{2\pi}{5}$ and legs of length 1. Thus the area is $5 \times \dfrac{1}{2}ab\sin C = \dfrac{5}{2}(1)(1)\sin\dfrac{2\pi}{5} = \boxed{\dfrac{5}{2}\sin\dfrac{2\pi}{5}}$, as required.

[8, 7 marks]

9. (a) Since f is a function of $\cos\theta$, it suffices to take θ in the range $0 \le \theta \le \pi$, since $\cos\theta$ is an even function and negative angles would therefore give the same outputs.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$f(\theta)$	1	1.54	2.17	3	5	7	7.83	8.46	9

Sketch:



(b) We solve $f(\theta) = 3$ to find the values of θ at which points of intersection occur.

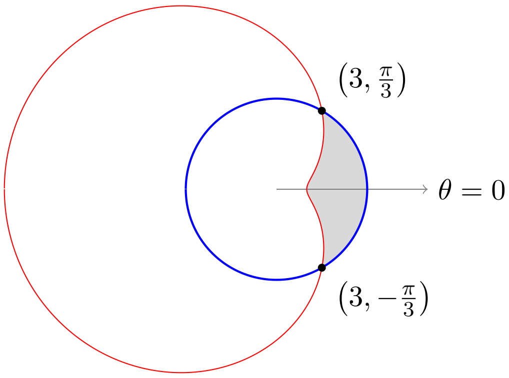
$$\begin{aligned}
&f(\theta) = 3\\
\implies &5 - 4\cos\theta = 3\\
\implies &\cos\theta = \frac{1}{2}\\
\implies &\theta_{\text{pv}} = \cos^{-1}\frac{1}{2} = \frac{\pi}{3}\\
\implies &\theta = 2n\pi \pm \frac{\pi}{3}, \qquad n \in \mathbb{Z}
\end{aligned}$$

Taking values of n other than $n = 0$ gives values outside the range $-\pi < \theta \leq \pi$, thus the only values of θ where intersection points occur are $\theta = \pm\frac{\pi}{3}$. Thus the points of intersection are $\boxed{P = \left(3, \frac{\pi}{3}\right) \text{ and } Q = \left(3, -\frac{\pi}{3}\right)}$.

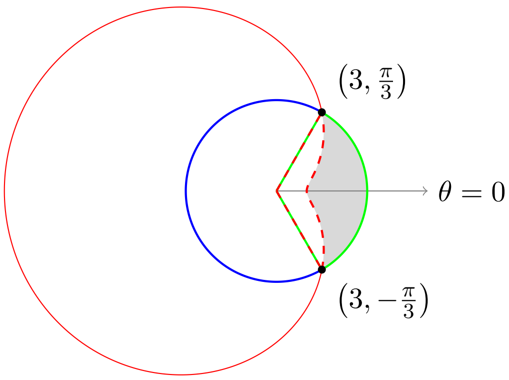
The curve $\mathcal{C}$ represents a $\boxed{\text{circle}}$, since it has a fixed radius of $r = 3$ independent of the angle θ.

(c) A line through P then the pole makes an angle of $\frac{\pi}{3} - \pi = -\frac{2\pi}{3}$. Thus we evaluate $f\left(-\frac{2\pi}{3}\right) = 7$, which represents the distance from the pole to the point R. Therefore the distance PR is the distance from the pole to P $(r = 3)$ plus the distance from the pole to R $(r = 7)$, i.e. $|PR| = \boxed{10}$.

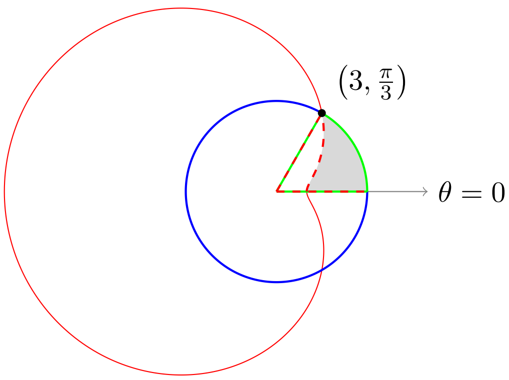
(d) The desired area is the following:



We know that in general, the area enclosed by the curve $r = r(\theta)$ between $r = a$ and $r = b$ is given by $\frac{1}{2}\int_a^b r^2 \,\mathrm{d}\theta$. Now our desired area is the area of the circle between $r = \frac{\pi}{3}$ (outlined in green) and $r = -\frac{\pi}{3}$, minus that of the curve $\mathcal{L}$ in that region (outlined in red, dashed):



Furthermore, since the region is symmetric in $\theta = 0$, we can simply evaluate the following enclosed region instead, then multiply by two:



Therefore:

$$\begin{aligned}
A &= \frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} (3)^2\,\mathrm{d}\theta - \frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} (5-4\cos\theta)^2\,\mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{3}} 9\,\mathrm{d}\theta - \int_0^{\frac{\pi}{3}} (25-40\cos\theta+16\cos^2\theta)\,\mathrm{d}\theta \qquad \text{(since the curve is symmetric in } \theta=0\text{)} \\
&= 9\theta - 25\theta + 40\sin\theta\Big|_0^{\frac{\pi}{3}} - 8\int_0^{\frac{\pi}{3}} (1+\cos 2\theta)\,\mathrm{d}\theta \\
&= 20\sqrt{3} - \frac{16\pi}{3} - 8\left(\theta + \frac{\sin 2\theta}{2}\right)\Bigg|_0^{\frac{\pi}{3}} \\
&= \boxed{18\sqrt{3} - 8\pi \text{ units}^2}
\end{aligned}$$

[3, 3, 3, 6 marks]

10. (a) Take $\mathbf{a} = \mathbf{i}+\mathbf{j}-3\mathbf{k}$ as the initial point, and $\vec{AB} = \mathbf{b}-\mathbf{a} = \mathbf{i}-3\mathbf{j}+4\mathbf{k}$ as its direction. Thus ℓ_1 has equation $\boxed{\mathbf{r} = \mathbf{i}+\mathbf{j}-3\mathbf{k}+\lambda(\mathbf{i}-3\mathbf{j}+4\mathbf{k})}$.

(b) Since the points A, B and C lie on Π_1, then the vectors $\vec{AB} = \mathbf{b}-\mathbf{a} = \mathbf{i}-3\mathbf{j}+4\mathbf{k}$ and $\vec{AC} = \mathbf{c}-\mathbf{a} = -\mathbf{j}+2\mathbf{k}$ lie in Π_1. Therefore we can define $\mathbf{n}_1 = \vec{AB}\times\vec{AC}$ to be the normal of Π_1, where

$$\mathbf{n}_1 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -3 & 4 \\ 0 & -1 & 2 \end{vmatrix} = \begin{vmatrix} -3 & 4 \\ -1 & 2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 4 \\ 0 & 2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & -3 \\ 0 & -1 \end{vmatrix}\mathbf{k}$$
$$= -2\mathbf{i}-2\mathbf{j}-\mathbf{k}$$

Therefore Π_1 has vector equation $\mathbf{r}\cdot\mathbf{n}_1 = \mathbf{a}\cdot\mathbf{n}_1$, i.e. $\mathbf{r}\cdot(2\mathbf{i}+2\mathbf{j}+\mathbf{k}) = (\mathbf{i}+\mathbf{j}-3\mathbf{k})\cdot(2\mathbf{i}+2\mathbf{j}+\mathbf{k})$, which simplifies to $\mathbf{r}\cdot(2\mathbf{i}+2\mathbf{j}+\mathbf{k}) = 1$, having the corresponding Cartesian equation $\boxed{2x+2y+z=1}$.

(c) Since Π_2 contains the points C and D, then the vector $\vec{CD} = \mathbf{d}-\mathbf{c} = -2\mathbf{i}+\mathbf{j}+\mathbf{k}$ lie on Π_2. Furthermore, since Π_2 does not intersect ℓ, then it must be parallel to ℓ, i.e. its direction vector $\vec{AB} = \mathbf{i}-3\mathbf{j}+4\mathbf{k}$ lies in the plane Π_2. Therefore we can define $\mathbf{n}_2 = \vec{AB}\times\vec{CD}$ to be the normal of Π_2, where

$$\mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -3 & 4 \\ -2 & 1 & 1 \end{vmatrix} = \begin{vmatrix} -3 & 4 \\ 1 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 4 \\ -2 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & -3 \\ -2 & 1 \end{vmatrix}\mathbf{k}$$
$$= -7\mathbf{i}-9\mathbf{j}-5\mathbf{k}$$

Therefore Π_2 has vector equation $\mathbf{r}\cdot\mathbf{n}_2 = \mathbf{c}\cdot\mathbf{n}_2$, i.e. $\mathbf{r}\cdot(7\mathbf{i}+9\mathbf{j}+5\mathbf{k}) = (\mathbf{i}-\mathbf{k})\cdot(7\mathbf{i}+9\mathbf{j}+5\mathbf{k})$, which simplifies to $\mathbf{r}\cdot(7\mathbf{i}+9\mathbf{j}+5\mathbf{k}) = 2$, having the corresponding Cartesian equation $\boxed{7x+9y+5z=2}$.

(d) The distance of a point X with position vector $\mathbf{x}$ from a plane $\Pi : \mathbf{r}\cdot\mathbf{n} = d$ is given by the formula $s = \left|\mathbf{x}\cdot\hat{\mathbf{n}} - \dfrac{d}{\|\mathbf{n}\|}\right|$. In the case of Π_2, we have $\|\mathbf{n}_2\| = \sqrt{7^2+9^2+5^2} = \sqrt{155}$, so $\hat{\mathbf{n}}_2 = \frac{7}{\sqrt{155}}\mathbf{i} + \frac{9}{\sqrt{155}}\mathbf{j} + \frac{5}{\sqrt{155}}\mathbf{k}$. Thus for the point A, we have:

$$\left|\mathbf{a}\cdot\hat{\mathbf{n}}_2 - \frac{2}{\|\mathbf{n}_2\|}\right| = \left|\frac{7}{\sqrt{155}} + \frac{9}{\sqrt{155}} - \frac{15}{\sqrt{155}} - \frac{2}{\sqrt{155}}\right| = \boxed{\frac{1}{\sqrt{155}} \text{ units}}$$

Similarly, for point B, we have

$$\left|\mathbf{b}\cdot\hat{\mathbf{n}}_2 - \frac{2}{\|\mathbf{n}_2\|}\right| = \left|\frac{14}{\sqrt{155}} - \frac{18}{\sqrt{155}} + \frac{5}{\sqrt{155}} - \frac{2}{\sqrt{155}}\right| = \boxed{\frac{1}{\sqrt{155}}\text{ units}}$$

(e) The angle θ between two planes whose normals are $\mathbf{n}_1$ and $\mathbf{n}_2$ is given by the formula $\cos\theta = \hat{\mathbf{n}}_1\cdot\hat{\mathbf{n}}_2$. In our case, we have $\hat{\mathbf{n}}_1 = \frac{\mathbf{n}_1}{\|\mathbf{n}_1\|} = \frac{2}{3}\mathbf{i}+\frac{2}{3}\mathbf{j}+\frac{1}{3}\mathbf{k}$ and $\hat{\mathbf{n}}_2 = \frac{7}{\sqrt{155}}\mathbf{i}+\frac{9}{\sqrt{155}}\mathbf{j}+\frac{5}{\sqrt{155}}\mathbf{k}$, thus

$$\begin{aligned}\cos\theta &= \hat{\mathbf{n}}_1\cdot\hat{\mathbf{n}}_2\\ &= \frac{14}{3\sqrt{155}} + \frac{18}{3\sqrt{155}} + \frac{5}{3\sqrt{155}}\\ &= \frac{37}{3\sqrt{115}}\\ \implies \theta &= \cos^{-1}\left(\frac{37}{3\sqrt{115}}\right) \approx \boxed{7.84^\circ}\end{aligned}$$

[2, 3, 4, 3, 3 marks]