A Level

20th April 2017 **3 hours**

Pure Mathematics

Paper II

20th April 2017

Question Paper

This paper consists of five pages and ten questions. Check to see if any pages are missing.

Answer any **SEVEN** questions. Each question carries **15** marks.

- Protractors and scientific calculators are permitted
- Graphical calculators are **not** permitted
- Check answers fully and present working fully as necessary
- Three hours are allocated for this test paper, utilise your time effectively

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Question Paper

1. (a) (i) Determine $\dfrac{\mathrm{d}}{\mathrm{d}x}\,[xy\cos x]$.

   (ii) Solve the differential equation

$$x\,\frac{\mathrm{d}y}{\mathrm{d}x} + y - xy\tan x = x^2,$$

   given that the curve it represents passes through the point $(\pi, 0)$.

   (b) Solve the differential equation

$$9\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 12\frac{\mathrm{d}y}{\mathrm{d}x} + 4y = 8x - 5e^{-x},$$

   given that when $x = 0$, $y = \dfrac{14}{5}$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{11}{5}$.

[7, 8 marks]

2. The *incomplete gamma function* Γ_n is given by the integral

$$\Gamma_n(x) = \int_0^x t^n\, e^{-t}\,\mathrm{d}t,$$

   where $n \geq 1$ is an integer.

   (a) Show that $I_n = n(n-1)I_{n-2} - x^{n-1}e^{-x}(x+n)$, where I_n is denoting $\Gamma_n(x)$.

   (b) Determine that $\Gamma_7(1) = 7! - 13\,700/e$.

   (c) The area bounded by the curve $y = x^{5/2}e^{-x/2}$ and the x-axis between $x = 0$ and $x = 2$ is rotated through $180°$ about the x-axis, so that one side of the solid generated is completely flat. Find the volume of the solid generated by this rotation.

[5, 4, 6 marks]

3. The real-valued function f is given by

$$f(x) = \frac{6x - x^2}{x^2 - 6x + 5}.$$

   (a) Determine the range of values of y in which no part of the curve $y = f(x)$ exists.

   (b) Using the result you obtained in part (a) or otherwise, determine the coordinates of any turning points to the curve $y = f(x)$. State also the equations of any asymptotes.

   (c) Sketch the curve $y = f(x)$, clearly indicating the turning points and asymptotes found in part (b), together with the points where the curve intersects coordinate axes.

   (d) Hence or otherwise, sketch the curve $y = \dfrac{1}{f(x)}$ on a separate diagram.

[4, 4, 4, 3 marks]

4. (a) Prove by induction that the sum of the first n positive integers is $n(n+1)/2$. Use this fact to prove, by induction, that

$$(1+2+3+\cdots+n)^2 = 1^3+2^3+3^3+\cdots+n^3$$

for any positive integer n.

(b) Prove by induction that

$$\sum_{r=1}^{2^n}\frac{1}{r} = \frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2^n-2}+\frac{1}{2^n-1}+\frac{1}{2^n} \geq 1+\frac{n}{2},$$

i.e. that the sum of the first 2^n terms of the *Harmonic series* $\displaystyle\sum_{r=1}^{\infty}\frac{1}{r}$ is at least $1+\dfrac{n}{2}$. Why is this enough to conclude that it diverges? [Hint: Consider the limit as $n\to\infty$]

[8, 7 marks]

5. (a) Show that the equation $\sin x + \cos x = x$ has a solution in the range $1 < x < 2$. Perform *two* iterations of the Newton-Raphson method to find an approximate value for this solution, taking $x_0 = 1$ as a first approximation. Give your working accurate to *four* decimal places.

[Note: Make sure that your calculator is set to *radians*.]

(b) The *Gaussian error function*, erf x, is defined by the integral

$$\operatorname{erf} x = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,\mathrm{d}t.$$

(i) Evaluate erf(0.6) by Simpson's Rule, taking an interval width of $h = 0.1$. Give your working accurate to *four* decimal places.

(ii) Determine the polynomial series expansion of $\frac{2}{\sqrt{\pi}}e^{-t^2}$, up to and including the term in t^6, and use it to approximate erf(0.6). Give your working accurate to *four* decimal places.

(iii) Given that $\operatorname{erf}(0.6) = 0.60385609$, state the %-error of the results you obtained in parts (i) and (ii). Which is the more accurate?

[7, 8 marks]

6. Consider the matrix $\mathbf{A} = \begin{pmatrix} -3 & 12 & 4 \\ -2 & 7 & 2 \\ 5 & a & b \end{pmatrix}$.

(a) Given that $(\mathbf{A}-\mathbf{I})(\mathbf{A}+2\mathbf{I}) = \mathbf{0}_{3\times3}$, where $\mathbf{I}$ and $\mathbf{0}_{3\times3}$ are the 3×3 identity and zero matrices respectively, determine the values of a and b. Using the given equation, or otherwise, determine $\mathbf{A}^{-1}$.

For the following question, ignore the values of a and b found in part (a).

(b) Let $\mathbf{B}$ be the matrix $\mathbf{A}$ above, with $b = 6$.

(i) Explain why the homogeneous equation $\mathbf{Bx} = \mathbf{0}_{3\times1}$ always has at least one solution, and that there is a unique value of a for which it has more than one solution.

(ii) Find this value of a, and determine the solutions of the equation when a takes on this value, in the form of a vector equation. What does this vector equation represent, geometrically?

[6, 9 marks]

7. (a) James McGraw's pirate ship is sinking. Being an avid reader, he quickly enters his quarters and starts packing his collection of 20 (distinct) books into two identical crates.

   (i) In how many different ways can the books be placed into the crates if he places 10 books in each crate?

   (ii) 12 of his books are by Robert Louis Stevenson. If the 20 books are randomly distributed in the two boxes (not necessarily 10 books in each crate as in part (i)), what is the probability that there are exactly 6 of R.L. Stevenson's books in each crate?

   (iii) James realises that two crates aren't enough, and fetches a third one, identical to the first two. After more packing, the last four books are left to pack: a collection of four poetry books by Virgil. They are randomly distributed among the three crates. What is the probability that each crate contains at least one of these books?

   (b) (i) Prove *Bayes' Theorem*: If $\mathrm{P}(B) \neq 0$, then

$$\mathrm{P}(A|B) = \frac{\mathrm{P}(B|A)\,\mathrm{P}(A)}{\mathrm{P}(B)}$$

   (ii) A group of students attend maths tutorials in Tarxien, and their tutor prepares tests for them on a regular basis. Being a fierce test-setter, the probability that a student passes a test is 45%.

   However, whether a student passes or not depends on how well-prepared they are. In fact, the probability that a student had studied for a test, given that he passed, is 96%. The problem is, that when a tutor announces a test, only 65% of the students study for it.

   What is the probability of passing a test, given that the student has studied for it?

[2, 3, 4, 3, 3 marks]

8. (a) Use De Moivre's Theorem to prove that

$$32\sin^6\theta \equiv 10 - 15\cos 2\theta + 6\cos 4\theta - \cos 6\theta,$$

   and determine a corresponding identity for $32\cos^6\theta$. Deduce that

$$8(\sin^6\theta + \cos^6\theta) \equiv 5 + 3\cos 4\theta,$$

   and hence or otherwise, evaluate the integral

$$\int_0^{64\pi} (\sin^6\theta + \cos^6\theta)^2 \,\mathrm{d}\theta.$$

   (b) Find the fifth roots of unity, and sketch them on an Argand diagram. Show that:

   (i) They can be written as $1, \omega, \omega^2, \omega^3$ and ω^4.

   (ii) Their sum is zero.

   [Hint: Consider the expression $(1-\omega)(1+\omega+\omega^2+\omega^3+\omega^4)$]

   (iii) The points they represent in the complex plane form the vertices of a pentagon whose area is $\frac{5}{2}\sin\frac{2\pi}{5}$.

[8, 7 marks]

9. A limaçon curve $\mathcal{L}$ is given by the polar equation $r = f(\theta)$, where $f(\theta) = 5 - 4\cos\theta$.
   (a) Use a suitable range of θ values to sketch the curve $r = f(\theta)$.
   (b) Determine the polar coordinates of the points P and Q, where the curve $\mathcal{L}$ intersects the curve $\mathcal{C}$, with equation $r = 3$. What is the curve $\mathcal{C}$?
   (c) A line passes through the point P, through the pole, and intersects the curve $\mathcal{L}$ at the point R. Determine the length of PR.
   (d) Determine the area of the region that lies within the curve $\mathcal{C}$, but outside the curve $\mathcal{L}$.

[3, 3, 3, 6 marks]

10. Relative to an origin O, the position vectors of the points A, B, C and D are

$$\mathbf{a} = \mathbf{i} + \mathbf{j} - 3\mathbf{k}, \qquad \mathbf{b} = 2\mathbf{i} - 2\mathbf{j} + \mathbf{k}, \qquad \mathbf{c} = \mathbf{i} - \mathbf{k}, \qquad \mathbf{d} = -\mathbf{i} + \mathbf{j}$$

respectively. Determine:
   (a) a vector equation of the line ℓ through the points A and B,
   (b) the Cartesian equation of the plane Π_1 containing the points A, B and C,
   (c) the Cartesian equation of the plane Π_2 containing the points C and D, which does not intersect the line ℓ,
   (d) the distance of the points A and B from Π_2,
   (e) the angle between Π_1 and Π_2.

[2, 3, 4, 3, 3 marks]