Pure Mathematics

Paper I

19th April 2017

Question Paper

This paper consists of four pages and ten questions. Check to see if any pages are missing.

Answer **ALL** questions. Each question carries **10** marks.

- Protractors and scientific calculators are permitted
- Graphical calculators are **not** permitted
- Check answers fully and present working fully as necessary
- Three hours are allocated for this test paper, utilise your time effectively

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Question Paper

1. A cardioid $\mathcal{C}$ is defined parametrically by the equations

$$x = \cos\theta(1 + 2\cos\theta) \qquad \text{and} \qquad y = \sin\theta(1 + 2\cos\theta),$$

where θ is a parameter in the range $-\pi \leq \theta \leq \pi$.

(a) Given a point (x, y) in the plane, let r represent its distance from the origin. Give a formula for r in terms of x and y, and use it show that each point on the curve $\mathcal{C}$ satisfies $r = 1 + 2\cos\theta$.
Hence or otherwise, find the value of θ which gives the point on $\mathcal{C}$ farthest from the origin.

(b) Determine the equations of the tangent and the normal to the cardioid $\mathcal{C}$ at the point where $\theta = \frac{\pi}{3}$.

(c) Show that the normal found in part (b) can be expressed as

$$r = 2\sqrt{7x^2 - 18x + 12} \qquad \text{or as} \qquad 3\sqrt{3}r = 2\sqrt{7y^2 - 2\sqrt{3}y + 12},$$

where r is the distance from the origin to a point (x, y) on the line in each case.
[You only need to prove one of these, but you may use both in the next part of the question.]

(d) Hence or otherwise, show that the point on the line closest to the origin is $\frac{1}{7}(9, \sqrt{3})$.

[3, 3, 2, 2 marks]

2. Solve the differential equation

$$(x^2 + 1)\frac{\mathrm{d}y}{\mathrm{d}x} = (x + 1)\cos y,$$

given that $y = \pi/6$ when $x = 0$.

[10 marks]

3. (a) Determine a vector equation of the line ℓ_1 passing through the points with position vectors $3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{i} + \mathbf{j} - 2\mathbf{k}$.

(b) The line ℓ_2 intersects and is perpendicular to ℓ_1, and passes through the position $\mathbf{p} = \mathbf{i} - 2\mathbf{j}$. Find its equation. Also, find the distance from the position $\mathbf{p}$ to ℓ_1.

[3, 7 marks]

4. The real-valued functions f and g are defined

$$f(x) = 5 - |x|, \quad \text{for } -5 \leq x \leq 5 \qquad \text{and} \qquad g(x) = \ln(x - 2).$$

(a) State the domain and range of f and g and sketch their graphs, clearly indicating any intercepts or asymptotes.

(b) Determine an expression for $g \circ f(x)$ and for an inverse function $(g \circ f)^{-1}(x)$.

(c) Carefully state the domain and range of $g \circ f(x)$.

[4, 2, 4 marks]

5. (a) Find a general solution of the equation $\sin m\theta = \cos n\theta$ for θ in radians.

   (b) The function $f(\theta)$ is given by

$$f(\theta) = 15\sin 2\theta + 8\cos 2\theta.$$

   Express $f(\theta)$ in the form $r\sin(2\theta + \alpha)$, and hence solve the equation $f(\theta) = 12$ for values of θ in the range $-180^\circ \leq \theta \leq 180^\circ$. Give your answers accurate to 2 d.p.s.

[5, 5 marks]

6. The complex numbers z_1 and z_2 are given by

$$z_1 = \frac{\alpha}{1+i} \qquad \text{and} \qquad z_2 = \frac{\beta}{1+2i},$$

where α and β are real numbers, and $z_1 + z_2 = i$.

   (a) Determine the values of α and β. Hence, express both z_1 and z_2 in the form $a + bi$.

   (b) Sketch the locus represented by $|z - z_1| = 4$ on an Argand diagram.

   (c) z_1 and z_2 are two of the roots of the fifth degree polynomial

$$p(x) = 3x^5 - x^4 + 9x^3 - 21x^2 + 36x - k,$$

   where k is a real number. Determine the constant k, and hence determine the other three roots of $p(x)$.

[3, 3, 4 marks]

7. (a) Evaluate the following integrals.

   (i) $\displaystyle\int_0^1 \frac{x^5}{\sqrt[3]{x^6+7}}\,\mathrm{d}x$ (ii) $\displaystyle\int_0^3 x^4 \ln\left(\frac{x}{3}\right)\mathrm{d}x$

   (b) Show that

$$\sqrt{\frac{1+x}{1-3x}} = 1 + 2x + 4x^2 + 10x^3 + \mathcal{O}(x^4).$$

   For what range of values of x does this expansion remain valid? By taking $x = 1/39$ in the expansion, approximate $\sqrt{10}$ and find the percentage error.

[3, 3, 4 marks]

8. (a) Find the number of four letter words, not necessarily meaningful, which can be formed out of the letters in the word

BLACKSAILS.

   (b) Jack Rackham and Anne Bonny go out for dinner with 3 other couples. They sit down randomly at a round table.

   (i) What is the probability that Jack and Anne sit next to each other?

   (ii) What is the probability that each person sits next to his partner?

[4, 6 marks]

9. The transformation $\mathbf{R}$ represents a rotation of $30°$ in the plane, whereas $\mathbf{P}$ represents a reflection in the line $y = 2x$.
   (a) Determine the 2×2 matrix representations of $\mathbf{R}$ and $\mathbf{P}$. What do the transformation matrices $\mathbf{RP}$ and $\mathbf{P}^{-1}\mathbf{R}^{-1}$ do? (Do not find them).
   (b) State the equation of the circle $\mathcal{C}$, centred at $(2, -3)$ with radius 2.
   (c) Determine the equation of $\mathcal{C}$ after being rotated by $30°$.

[4, 2, 4 marks]

10. The points $(2, 1)$ and $(3, 4)$ are two diagonally opposite vertices of a square.
   (a) Find the coordinates of its other two vertices.
   (b) Show that the equation of the inscribed circle of this square is $4(x^2 + y^2) + 45 = 20(x + y)$.

[6, 4 marks]

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