MATHEMATICS TUTORIALS
ĦAL TARXIEN

A Level – First Year

17th April 2016 **3 hours**

Pure Mathematics

Paper I

17th April 2016

Solutions

If any errors are found in these solutions, please contact the author by email on `luke@maths.com.mt`, or call 79 000 126.

Solutions

1. (a) $\dfrac{3x^2-2x+11}{(x^2+1)(x-4)} \equiv \dfrac{Ax+B}{x^2+1} + \dfrac{C}{x-4}$.

For C, multiply throughout by $(x-4) \implies \dfrac{3x^2-2x+11}{x^2+1} \equiv \dfrac{(Ax+B)(x-4)}{x^2+1} + C$

Now let $x = 4 \implies \dfrac{3(4)^2-2(4)+11}{(4)^2+1} = 0 + C$, Hence $C = 3$.

Now, for A and B:

$$\begin{aligned}\frac{3x^2-2x+11}{(x^2+1)(x-4)} &\equiv \frac{Ax+B}{x^2+1} + \frac{3}{x-4}\\ &\equiv \frac{(Ax+B)(x-4)+3(x^2+1)}{(x^2+1)(x-4)}\\ &\equiv \frac{(3+A)x^2+(B-4A)x+3-4B}{(x^2+1)(x-4)}\end{aligned}$$

$$\implies 3x^2-2x+11 \equiv (3+A)x^2+(B-4A)x+3-4B$$

Since we have two identical polynomials in x, their coefficients must be equal. Comparing the coefficients of x^2, we get $3 = 3 + A$, so $A = 0$. Now comparing the coefficients of x^0 (i.e. the constant terms), we have $11 = 3 - 4B \implies 8 = -4B$, so $B = -2$.

$$\boxed{\therefore \frac{3x^2-2x+11}{(x^2+1)(x-4)} \equiv \frac{3}{x-4} - \frac{2}{x^2+1}}$$

(b) $(x^2+1)(x-4)\dfrac{\mathrm{d}y}{\mathrm{d}x} = \sec^4 y\,(3x^2-2x+11)$

$$\implies \frac{1}{\sec^4 y}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3x^2-2x+11}{(x^2+1)(x-4)}$$

Integrating both sides with respect to x:

$$\implies \int \cos^4 y \frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x = \int \frac{3x^2-2x+11}{(x^2+1)(x-4)}\,\mathrm{d}x$$

$$\implies \int \cos^4 y \frac{\mathrm{d}y}{\cancel{\mathrm{d}x}}\,\cancel{\mathrm{d}x} = \int \left(\frac{3}{x-4} - \frac{2}{x^2+1}\right)\mathrm{d}x \qquad \text{(by part (a) above)}$$

$$\implies \int \cos^4 y\,\mathrm{d}y = 3\int \frac{1}{x-4}\,\mathrm{d}x - 2\int \frac{1}{x^2+1}\,\mathrm{d}x$$

$$\implies \int (\cos^2 y)^2\,\mathrm{d}y = 3\ln|x-4| - 2\tan^{-1} x \qquad (*)$$

Now, consider the left-hand side:

$$\begin{aligned}\int (\cos^2 y)^2\,\mathrm{d}y &= \int \left(\frac{1}{2}(\cos 2y + 1)\right)^2 \mathrm{d}y = \frac{1}{4}\int \left(\cos^2 2y + 2\cos 2y + 1\right)\mathrm{d}y\\ &= \frac{1}{4}\int \cos^2 2y\,\mathrm{d}y + \frac{1}{2}\int \cos 2y\,\mathrm{d}y + \frac{1}{4}\int \mathrm{d}y\\ &= \frac{1}{4}\int \frac{1}{2}(\cos 4y + 1)\,\mathrm{d}y + \frac{\sin 2y}{4} + \frac{y}{4} = \frac{\sin 4y}{32} + \frac{y}{8} + \frac{\sin 2y}{4} + \frac{y}{4} + c\end{aligned}$$

Substituting back in $(*)$, we have

$$\frac{\sin 4y}{32} + \frac{y}{8} + \frac{\sin 2y}{4} + \frac{y}{4} + c = 3\ln|x-4| - 2\tan^{-1} x$$
$$\implies \sin 4y + 8\sin 2y + 12y + c = 96\ln|x-4| - 64\tan^{-1} x$$

as the *general solution*. Now we can obtain the particular solution by determining the arbitrary constant c, since $y = 0$ when $x = 5$:

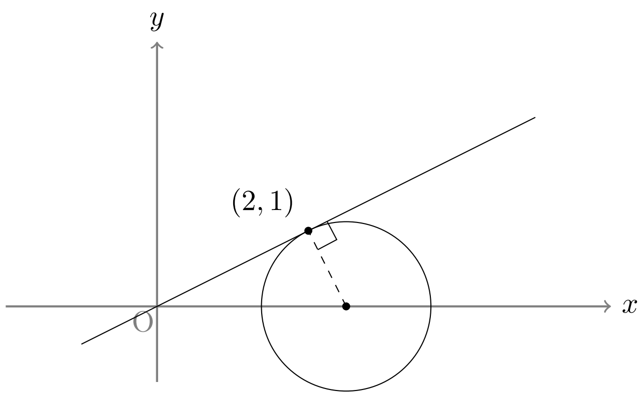
$$\left.\begin{matrix} x = 5 \\ y = 0 \end{matrix}\right\} \implies \sin 0 + 8\sin 0 + 12(0) + c = 96\ln 1 - 64\tan^{-1} 5$$
$$\implies c = -64\tan^{-1} 5.$$

Therefore the *particular solution* is

$$\boxed{\sin 4y + 8\sin 2y + 12y - 64\tan^{-1} 5 = 96\ln|x-4| - 64\tan^{-1} x}$$

[3, 7 marks]

2. (a) We have the following scenario:



We need to find the equation of a circle – this involves finding its centre (a, b) and its radius r, then substituting these into the equation $(x-a)^2 + (y-b)^2 = r^2$.

Let us start with the centre. Since the circle is centred at the origin, then the normal to the circle at $(2, 1)$ (i.e. the radius line in the diagram) intersects the x-axis at the centre. Now the given tangent has gradient $m = \dfrac{1-0}{2-0} = \dfrac{1}{2}$, so the normal at the same point has gradient $m' = -\dfrac{1}{m} = -2$.

Hence the equation of the normal is $y - 1 = -2(x-2) \implies y = 5 - 2x$. Now that we have this equation, substituting $y = 0$ will give us the x-intercept. $y = 0 \implies 0 = 5 - 2x \implies 2x = 5 \implies x = \frac{5}{2}$. Thus the coordinates of the centre are $(\frac{5}{2}, 0)$.

Now the radius is easy to find – we know the centre, and we know a point on the circumference. Thus r is simply the distance between these two points:

$$r = \sqrt{\left(2 - \frac{5}{2}\right)^2 + (1-0)^2} = \frac{\sqrt{5}}{2}.$$

Therefore the equation of the circle $\mathscr{C}$ is $\boxed{\left(x - \frac{5}{2}\right)^2 + y^2 = \frac{5}{4}}$.

(b) If $\ell_2 : 2y = 5 - x$ is a tangent to the circle $\mathscr{C}$, then they intersect at only one point. We show this by solving their equations simultaneously. We have the system

$$\begin{cases} \left(x - \frac{5}{2}\right)^2 + y^2 = \frac{5}{4} & (1) \\ 2y = 5 - x & (2) \end{cases}$$

From (2), we get that $y = (5 - x)/2$. Substituting this in (1) gives

$$\begin{aligned} &\left(x - \frac{5}{2}\right)^2 + \left(\frac{5 - x}{2}\right)^2 = \frac{5}{4} \\ \implies &x^2 - 5x + \frac{25}{4} + \frac{x^2 - 10x + 25}{4} = \frac{5}{4} \\ \implies &5x^2 - 30x + 50 = 5 \\ \implies &x^2 - 6x + 9 = 0 \\ \implies &(x - 3)^2 = 0 \\ \implies &x = 3 \quad \text{(twice)} \end{aligned}$$

i.e. The circle $\mathscr{C}$ and ℓ_2 intersect only at one point, the point where $x = 3$. □

(c) We solve the system

$$\begin{cases} \ell_1 : 2y = x & (1) \\ \ell_2 : 2y = 5 - x & (2) \end{cases}$$

to find the point of intersection. $(1) - (2)$ gives $0 = 2x - 5$, so $x = \frac{5}{2}$. Substituting into (1), we get $y = \frac{x}{2} = \frac{5}{4}$. Hence ℓ_1 and ℓ_2 intersect at $\boxed{\left(\frac{5}{2}, \frac{5}{4}\right)}$.

Now to verify that "*tangents from the same point are equal in length*", we show that the distance from the intersection point $\left(\frac{5}{2}, \frac{5}{4}\right)$ to the point of contact with the circle is equal for both tangents. Let d_1, d_2 represent these distances for ℓ_1 and ℓ_2 respectively.

Since ℓ_1 intersects the circle at $(2, 1)$, $d_1 = \sqrt{\left(\frac{5}{2} - 2\right)^2 + \left(\frac{5}{4} - 1\right)^2} = \frac{\sqrt{5}}{4}$.

Now recall from part (b) that ℓ_2 intersects the circle at the point where $x = 3$, so the corresponding y-coordinate is given by substitution in ℓ_2's equation: $y = (5 - 3)/2 = 1$. Hence the point of contact is $(3, 1)$. Thus the distance

$$d_2 = \sqrt{\left(\frac{5}{2} - 3\right)^2 + \left(\frac{5}{4} - 1\right)^2} = \frac{\sqrt{5}}{4} = d_1,$$

as required. □

[5, 2, 3 marks]

3. (a) (i) $f(x) \equiv \cos x + \sqrt{3} \sin x \equiv \lambda \cos(x - \alpha)$.
$\equiv \lambda \cos x \cos \alpha + \sin x \sin \alpha$ (compound angle identity for cos).

Since our claim is that these are identical for all x, then the coefficients of $\cos x$ and $\sin x$ must be equal. Thus we can compare:

$$\begin{cases} \cos x : 1 = \lambda \cos \alpha & (1) \\ \sin x : \sqrt{3} = \lambda \sin \alpha & (2) \end{cases}$$

$(2) \div (1) \implies \tan\alpha = \sqrt{3}$, so $\alpha = \tan^{-1}\sqrt{3} = \pi/3$.

$(1)^2 + (2)^2 \implies 1^2 + (\sqrt{3})^2 = \lambda^2\cos^2\alpha + \lambda^2\sin^2\alpha \implies 4 = \lambda^2$, so $\lambda = 2$.

Hence $\boxed{f(x) = 2\cos\left(x - \frac{\pi}{3}\right)}$.

(ii) For $f(x)$ intercepts, we substitute $x = 0$: $f(0) = 2\cos\frac{\pi}{3} = 1$, thus the f-intercept is $f = 1$. Now for x-intercepts, we set $f = 0$, i.e. we solve

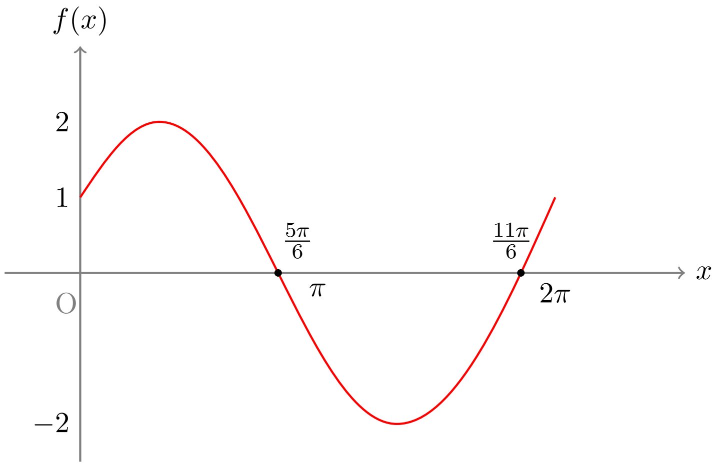
$$\begin{aligned} & f(x) = 0 \\ \implies & 2\cos\left(x - \frac{\pi}{3}\right) = 0 \\ \implies & \cos\left(x - \frac{\pi}{3}\right) = 0 \\ \implies & \left(x - \frac{\pi}{3}\right)_{\text{pv}^*} = \cos^{-1} 0 = \frac{\pi}{2} \\ \implies & x - \frac{\pi}{3} = 2n\pi \pm \frac{\pi}{2} \qquad n \in \mathbb{Z} \end{aligned}$$

Now substituting values for n:

$$n = 0 \implies x = \frac{5\pi}{6}, \qquad n = 1 \implies x = \frac{11\pi}{6}$$

Thus the x-intercepts in the range $[0, 2\pi]$ are $x = \frac{5\pi}{6}$, $x = \frac{11\pi}{6}$.

Sketch:



(iii) Clearly, from the sketch above,

$$\begin{aligned} & -2 \le f(x) \le 2 \\ \iff & -1 \le f(x) + 1 \le 3 \\ \iff & -1 \ge \frac{1}{f(x)+1} \quad \text{or} \quad \frac{1}{f(x)+1} \ge \frac{1}{3} \end{aligned}$$

Hence $\boxed{\min\left(\frac{1}{f(x)+1}\right) = \frac{1}{3}}$.

*This indicates that we are dealing only with the principal value of the angle, not all its possible values.

Now to find the value of x for which the minimum occurs, we want

$$\begin{aligned} &\frac{1}{f(x)+1} = \frac{1}{3} \\ \implies &f(x)+1 = 3 \\ \implies &f(x) = 2 \\ \implies &2\cos\left(x-\frac{\pi}{3}\right) \\ \implies &\left(x-\frac{\pi}{3}\right)_{\text{pv}} = \cos^{-1} 1 = 0 \\ &\boxed{\therefore x = \frac{\pi}{3}} \end{aligned}$$

Taking values of n in the general solution is skipped here, since there is only one value of x in the range $[0, 2\pi]$, so it must be the one from the principal value.

(b) To solve such equations, we make use of the *sum-to-product* trigonometric identities.

$$\begin{aligned} &\sin\theta + \sin 3\theta + \sin 5\theta = 0 \\ \implies &\sin\theta + \sin 5\theta + \sin 3\theta = 0 \\ \implies &2\sin\left(\frac{\theta+5\theta}{2}\right)\cos\left(\frac{\theta-5\theta}{2}\right) + \sin 3\theta = 0 \\ \implies &2\sin 3\theta\cos 2\theta + \sin 3\theta = 0 \\ \implies &\sin 3\theta(2\cos 2\theta + 1) = 0 \\ \implies &\sin 3\theta = 0 \quad \text{or} \quad 2\cos 2\theta + 1 = 0 \end{aligned}$$

We find the values of θ satisfying both cases:

$$\begin{aligned} &\sin 3\theta = 0 \\ \implies &(3\theta)_{\text{pv}} = \sin^{-1} 0 = 0 \\ \implies &3\theta = 180^\circ n, \quad n \in \mathbb{Z} \\ \implies &\theta = 60^\circ n, \quad n \in \mathbb{Z} \end{aligned}$$

Taking $n = 1, 2, 3, 4, 5$, we get the following values of θ in the range $0 < \theta < 360^\circ$.

$$\theta = \{60^\circ, 120^\circ, 180^\circ, 240^\circ, 300^\circ\}$$

$$\begin{aligned} &2\cos 2\theta + 1 = 0 \\ \implies &\cos 2\theta = -\frac{1}{2} \\ \implies &(2\theta)_{\text{pv}} = \cos^{-1}\left(-\frac{1}{2}\right) = 120^\circ \\ \implies &2\theta = 360^\circ n \pm 120^\circ, \quad n \in \mathbb{Z} \\ \implies &\theta = 180^\circ n \pm 60^\circ, \quad n \in \mathbb{Z} \end{aligned}$$

Taking $n = 0, 1, 2$, we get the following values of θ in the range $0 < \theta < 360^\circ$.

$$\theta = \{60^\circ, 120^\circ, 240^\circ, 300^\circ\}$$

Thus combining the results from both cases, we get

$$\boxed{\theta = \{60^\circ, 120^\circ, 180^\circ, 240^\circ, 300^\circ\}}$$

[6, 4 marks]

4. (a) $\mathbf{I}_* = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\Bigg|_{\theta=90^\circ} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

(i) $\mathbf{I}_*{}^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -\mathbf{I}$, as required.

(ii) To prove: $(\mathbf{I}\cos\theta + \mathbf{I}_*\sin\theta)^n = \mathbf{I}\cos n\theta + \mathbf{I}_*\sin n\theta$, by induction on n.

Proof. For the base case, take $n = 1$. The equation is obvious.

Hypothesis: Suppose that for $n = k \in \mathbb{N}$, we have

$$(\mathbf{I}\cos\theta + \mathbf{I}_*\sin\theta)^k = \mathbf{I}\cos k\theta + \mathbf{I}_*\sin k\theta.$$

Now for the inductive step.

$$\begin{aligned}
(\mathbf{I}\cos\theta + \mathbf{I}_*\sin\theta)^{k+1} &= (\mathbf{I}\cos\theta + \mathbf{I}_*\sin\theta)^1(\mathbf{I}\cos\theta + \mathbf{I}_*\sin\theta)^k \\
&= (\mathbf{I}\cos\theta + \mathbf{I}_*\sin\theta)(\mathbf{I}\cos k\theta + \mathbf{I}_*\sin k\theta) \qquad \text{(by the hypothesis)} \\
&= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos k\theta & -\sin k\theta \\ \sin k\theta & \cos k\theta \end{pmatrix} \\
&= \begin{pmatrix} \cos\theta\cos k\theta - \sin\theta\sin k\theta & -\sin k\theta\cos\theta - \cos k\theta\sin\theta \\ \sin\theta\cos k\theta + \cos\theta\sin k\theta & \cos k\theta\cos\theta - \sin k\theta\sin\theta \end{pmatrix} \\
&= \begin{pmatrix} \cos(k+1)\theta & -\sin(k+1)\theta \\ \sin(k+1)\theta & \cos(k+1)\theta \end{pmatrix} \\
&= \mathbf{I}\cos(k+1)\theta + \mathbf{I}_*\sin(k+1)\theta,
\end{aligned}$$

i.e. the statement holds for $n = k + 1$. □

(b) Let $z = r_1(\cos\theta_1 + i\sin\theta_1)$ and $w = r_2(\cos\theta_2 + i\sin\theta_2)$. Then

$$\begin{aligned}
\frac{z}{w} &= \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} \\
&= \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} \cdot \frac{\cos\theta_2 - i\sin\theta_2}{\cos\theta_2 - i\sin\theta_2} \\
&= \frac{r_1(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2))}{r_2(\cos^2\theta_2 + \sin^2\theta_2)} \\
&= \frac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2))
\end{aligned}$$

Hence $\left|\frac{z}{w}\right| = \frac{r_1}{r_2} = \frac{|z|}{|w|}$. Similarly,

$$\begin{aligned}
zw &= r_1(\cos\theta_1 + i\sin\theta_1) \cdot r_2(\cos\theta_2 + i\sin\theta_2) \\
&= r_1 r_2(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)) \\
&= r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))
\end{aligned}$$

Hence $\arg zw = \theta_1 + \theta_2 = \arg z + \arg w$. □

Multiplication by i rotates complex numbers in the Argand diagram by 90°, since given a complex number z, $\arg zi = \arg z + \arg i = \arg z + 90°$. This, in fact, explains the behaviour of the transformation matrix $\mathbf{I}_*$, since multiplying a vector by it is equivalent to multiplying by i.

[6, 4 marks]

5. (a) (i)

$$\begin{aligned}
x &= e^t\cos t \\
\implies \frac{\mathrm{d}x}{\mathrm{d}t} &= e^t\cos t - e^t\sin t \\
&= x - y
\end{aligned}
\qquad\qquad
\begin{aligned}
y &= e^t\sin t \\
\implies \frac{\mathrm{d}y}{\mathrm{d}t} &= e^t\sin t + e^t\cos t \\
&= y + x
\end{aligned}$$

By the chain rule, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}x} = (y+x)\frac{1}{x-y}$, so $\boxed{\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x+y}{x-y}}$.

(ii)
$$4x^2 - 3xy + 5y^3 + 2\cos y \sin x = 7$$
$$\implies \frac{\mathrm{d}}{\mathrm{d}x}(4x^2 - 3xy + 5y^3 + 2\cos y \sin x) = \frac{\mathrm{d}}{\mathrm{d}x}(7)$$
$$\implies 8x - 3y - 3x\frac{\mathrm{d}y}{\mathrm{d}x} + 15y^2\frac{\mathrm{d}y}{\mathrm{d}x} + 2\cos y\cos x - 2\sin y \sin x\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$\implies \frac{\mathrm{d}y}{\mathrm{d}x}(15y^2 - 3x - 2\sin y\sin x) = 3y - 8x - 2\cos y\cos x$$
$$\therefore \boxed{\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3y - 8x - 2\cos y\cos x}{15y^2 - 3x - 2\sin y\sin x}}$$

(b) $y = \exp(\sin^2 x - \cos x)$
$$\implies \frac{\mathrm{d}y}{\mathrm{d}x} = (2\sin x\cos x + \sin x)\exp(\sin^2 x - \cos x)$$
$$\implies \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=\frac{\pi}{2}} = \left(2\sin\frac{\pi}{2}\cos\frac{\pi}{2} + \sin\frac{\pi}{2}\right)\exp\left(\sin^2\frac{\pi}{2} - \cos\frac{\pi}{2}\right) = \exp 1 = e$$

Now, at $x = \frac{\pi}{2}$, $y = \exp(\sin^2\frac{\pi}{2} - \cos\frac{\pi}{2}) = \exp 1 = e$. Thus the equation of the tangent is
$$\boxed{y - e = e\left(x - \frac{\pi}{2}\right)}$$

(c) $x + y = 15$, we want to find $\max xy^2$.

$$\text{Set } \frac{\mathrm{d}}{\mathrm{d}y}(xy^2) = 0 \qquad \text{(for critical values)}$$
$$\implies \frac{\mathrm{d}}{\mathrm{d}y}((15-y)y^2) = 0 \qquad \text{(since } x + y = 15\text{)}$$
$$\implies \frac{\mathrm{d}}{\mathrm{d}y}(15y^2 - y^3) = 0$$
$$\implies 30y - 3y^2 = 0$$
$$\implies y(10 - y) = 0$$
$$\implies y = 0 \ ※ \quad \text{or} \quad y = 10$$

Thus the maximum occurs when $y = 10$, so $\max xy^2 = (15-10)10^2 = \boxed{500}$.

[4, 3, 3 marks]

6. (a) (i) $p(x) = 3 - 2x - x^2 = -(x^2 + 2x - 3) = -[(x+1)^2 - 3 - 1] = -[(x+1)^2 - 4] = 4 - (x+1)^2$.
Thus by completing the square, we obtained $\boxed{a = 4,\ b = 1}$.

(ii) $\displaystyle\int_0^1 \sqrt{3 - 2x - x^2}\,\mathrm{d}x$

$$= \int_0^1 \sqrt{p(x)}\,\mathrm{d}x \qquad \text{Let } x + 1 = 2\sin u \quad (*)$$
$$= \int_0^1 \sqrt{4 - (x+1)^2}\,\mathrm{d}x \qquad \implies \frac{\mathrm{d}x}{\mathrm{d}u} = 2\cos u$$
$$= \int_{x=0}^{x=1} \sqrt{4 - 4\sin^2 u}\,\mathrm{d}x \qquad x = 0 \implies u = \sin^{-1}\tfrac{0+1}{2} = \tfrac{\pi}{6}, \quad \text{by } (*)$$
$$x = 1 \implies u = \sin^{-1}\tfrac{1+1}{2} = \tfrac{\pi}{2}$$
$$= 2\int_{u=\frac{\pi}{6}}^{u=\frac{\pi}{2}} \sqrt{1 - \sin^2 u}\,\frac{\mathrm{d}x}{\mathrm{d}u}\,\mathrm{d}u$$

$$= 2\int_{u=\frac{\pi}{6}}^{u=\frac{\pi}{2}} \sqrt{\cos^2 u}\ (2\cos u)\,\mathrm{d}u$$
$$= 4\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cos^2 u\,\mathrm{d}u$$
$$= 4\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(\frac{1}{2}(\cos 2u + 1)\right)\,\mathrm{d}u$$
$$= 2\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (\cos 2u + 1)\,\mathrm{d}u$$
$$= 2\left(\frac{\sin 2u}{2} + u\right)\Bigg|_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$
$$= \boxed{\frac{4\pi - 3\sqrt{3}}{6}}$$

(b) Let $I = \int e^{\alpha\theta}\cos\beta\theta\,\mathrm{d}\theta$

$$= uv - \int v\,\mathrm{d}u \qquad \begin{array}{ll} u = e^{\alpha\theta} & \mathrm{d}v = \cos\beta\theta\,\mathrm{d}\theta \\ \Rightarrow \mathrm{d}u = \alpha e^{\alpha\theta}\,\mathrm{d}\theta & \Rightarrow v = \dfrac{\sin\beta\theta}{\beta} \end{array}$$
$$= \frac{e^{\alpha\theta}\sin\beta\theta}{\beta} - \frac{\alpha}{\beta}\int e^{\alpha\theta}\sin\beta\theta\,\mathrm{d}\theta$$
$$= \frac{e^{\alpha\theta}\sin\beta\theta}{\beta} - \frac{\alpha}{\beta}\left(wx - \int x\,\mathrm{d}w\right) \qquad \begin{array}{ll} w = e^{\alpha\theta} & \mathrm{d}x = \sin\beta\theta\,\mathrm{d}\theta \\ \Rightarrow \mathrm{d}w = \alpha e^{\alpha\theta}\,\mathrm{d}\theta & \Rightarrow x = -\dfrac{\cos\beta\theta}{\beta} \end{array}$$
$$= \frac{e^{\alpha\theta}\sin\beta\theta}{\beta} - \frac{\alpha}{\beta}\left(-\frac{e^{\alpha\theta}\cos\beta\theta}{\beta} + \frac{\alpha}{\beta}\int e^{\alpha\theta}\cos\beta\theta\,\mathrm{d}\theta\right)$$
$$\implies I = \frac{\beta e^{\alpha\theta}\sin\beta\theta + \alpha e^{\alpha\theta}\cos\beta\theta}{\beta^2} - \frac{\alpha^2}{\beta^2}I$$
$$\implies \left(1 + \frac{\alpha^2}{\beta^2}\right)I = \frac{e^{\alpha\theta}(\alpha\cos\beta\theta + \beta\sin\beta\theta)}{\beta^2}$$
$$\implies I = \frac{e^{\alpha\theta}(\alpha\cos\beta\theta + \beta\sin\beta\theta)}{\beta^2\left(1 + \frac{\alpha^2}{\beta^2}\right)}$$

Therefore $\int e^{\alpha\theta}\cos\beta\theta\,\mathrm{d}\theta = \boxed{\dfrac{e^{\alpha\theta}}{\alpha^2+\beta^2}(\alpha\cos\beta\theta + \beta\sin\beta\theta)}$.

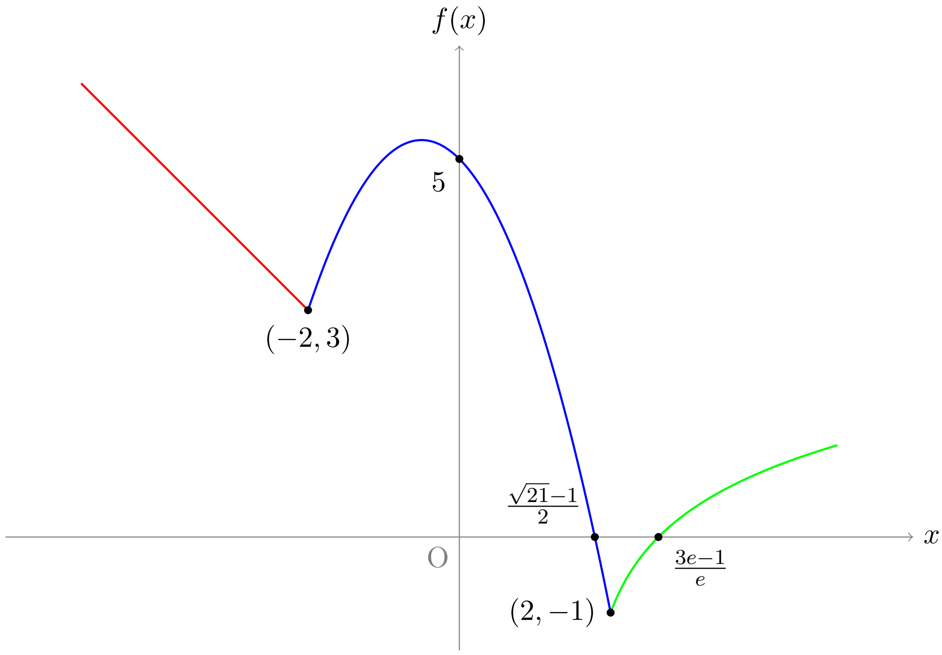
[5, 5 marks]

7. (a) If f is well-defined at $x = -2$, then $f(-2) = 1 + 2 = 3$ (since $f(x) = 1 - x$ for $x \leq -2$), must also equal $f(-2) = a + 2 - (-2)^2 = a - 2$ (since $f(x) = a - x - x^2$ for $-2 \leq x \leq 2$). In other words, we want the outputs at the endpoints to be the same (so that the curves are connected), thus we want $3 = a - 2 \implies a = 5$.

Similarly at $x = 2$, we want $f(2) = 5 - 2 - 2^2 = -1$ (since $f(x) = 5 - x - x^2$ for $-2 \leq x \leq 2$) to equal $f(2) = \ln(2 + b)$ (since $f(x) = \ln(x + b)$ for $x \geq 2$). Thus we equate and solve $-1 = \ln(2 + b) \implies e^{-1} = 2 + b \implies b = e^{-1} - 2$.

Therefore $\boxed{a = 5,\ b = e^{-1} - 2}$.

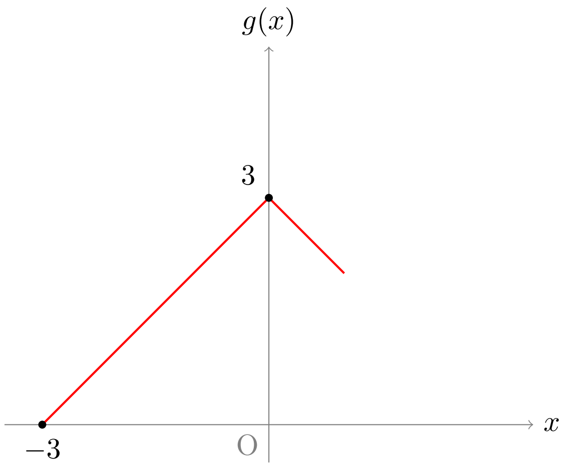
(b) Sketch:



(Note: Intercepts are determined in the usual way, evaluating $f(0)$ for the f-intercept, and solving $f(x) = 0$ for the different parts of the function for the x-intercepts).

Clearly f is defined for all real numbers, so the domain $\boxed{\text{Dom } f = \mathbb{R}}$, and since no part of the curve lies below the line $f = -1$, the range $\boxed{\text{Ran } f = [-1, \infty)}$.

(c) Sketch:



Since all of the curve lies between $g = 3$ and $g = 0$, both *excluded*, then $\boxed{\text{Ran } g = (0, 3)}$.

Now, to show that there are no values of x for which $f(x) = g(x)$, let us attempt to solve this equation on different parts of the common domain. When $-3 < x \leq -2$, the equation $f(x) = g(x)$ becomes

$$x - 1 = 3 + x,$$

where $3 - |x|$ becomes $3 + x$ since x is always negative in this part of the domain, so $|x| = -x$. The equation above clearly has no solutions. Now when $-2 \leq x < 1$, the equation $f(x) = g(x)$ becomes

$$5 - x - x^2 = 3 - |x|.$$

With a little more work, we can show that this equation has no solutions either:

$$
\begin{aligned}
& 5 - x - x^2 = 3 - |x| \\
\Longrightarrow\ & |x| = x^2 + x - 2 \\
\Longrightarrow\ & \pm x = x^2 + x - 2, \qquad + \text{ when } x \geq 0, \ - \text{ when } x < 0 \\
\Longrightarrow\ & x^2 + x \mp x - 2 = 0, \qquad - \text{ when } x \geq 0, \ + \text{ when } x < 0 \\
\Longrightarrow\ & x^2 + (1 \mp 1)x - 2 = 0 \\
\Longrightarrow\ & x^2 - 2 = 0 \text{ for when } x \geq 0 \qquad \text{or} \qquad x^2 + 2x - 2 = 0 \text{ for when } x < 0 \\
\Longrightarrow\ & x = \sqrt{2} \ \text{※} \ \text{(not in the domain)} \qquad \Longrightarrow x = -1 - \sqrt{3} \ \text{※} \ \text{(not in the domain)} \\
\text{or}\ & x = -\sqrt{2} \ \text{※} \ (\text{not } \geq 0) \qquad \text{or} \quad x = \sqrt{3} - 1 \ \text{※} \ (\text{not } < 0)
\end{aligned}
$$

Thus there are no values of x for which $f(x) = g(x)$. □

[2, 3, 2, 3 marks]

8. (a) All the letters here are distinct, apart from the two 'E's. Suppose that our word has at most one 'E' for now, i.e. we are choosing four distinct elements from the set

$$\{\text{P, R, E, S, I, D, N, T}\}.$$

We can do this in $\binom{8}{4}$ ways, and since different orderings of the same four letters are considered different words, we have $\binom{8}{4} \times 4!$ words of this form. (Alternatively, this number is ${}^8\text{P}_4$).

Now the only remaining case is that in which we have exactly two 'E's. Let construct such words in the following way:

___ ___ ___ ___

We choose where to put the 'E's first. This can be done in $\binom{4}{2}$ ways. We then have seven letters (excluding 'E') left to choose from. We have $\binom{7}{1}$ options for one of the two remaining slots, and $\binom{6}{1}$ for the final slot.

Hence the solution is $\binom{8}{4} \times 4! + \binom{4}{2} \times \binom{7}{1} \times \binom{6}{1} = \boxed{1932 \text{ words}}$.

(b) As a product of primes, $7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 = 7 \times 3 \times 2 \times 5 \times 2 \times 2 \times 3 \times 2 = 2^4 \times 3^2 \times 5 \times 7$.

Every factor of 7! must therefore be of the form $2^a \times 3^b \times 5^c \times 7^d$, where $a \in \{0, 1, 2, 3, 4\}$, $b \in \{0, 1, 2\}$, $c \in \{0, 1\}$ and $d \in \{0, 1\}$. So clearly we have 5 choices for the value of a, 3 choices for b, and 2 for both c and d. Therefore 7! has $5 \times 3 \times 2 \times 2 = \boxed{60 \text{ factors}}$.

(c) Let H be the event of tossing a head on a fair coin, and let D be the event of rolling a 4 or a 6 on a fair die. Clearly, $\text{P}(H) = \frac{1}{2}$ and $\text{P}(D) = \frac{1}{3}$.

Now let H_n be the event of tossing a heads on the nth go. Clearly, to even get to the nth go, the previous $2(n-1)$ must have all been unwanted outcomes (otherwise, Donald would have stopped). Thus the probability of H_n is the probability of $2(n-1)$ successive unwanted outcomes, followed by the final heads toss:

$$
\begin{aligned}
\text{P}(H_n) &= \text{P}\left(\left(\bigcap_{r=1}^{n-1} (\bar{H} \cap \bar{D})\right) \cap H\right) = \text{P}\left(\bigcap_{r=1}^{n-1} (\bar{H} \cap \bar{D})\right) \times \text{P}(H) \\
&= \underbrace{\text{P}(\bar{H} \cap \bar{D}) \times \text{P}(\bar{H} \cap \bar{D}) \times \cdots \times \text{P}(\bar{H} \cap \bar{D})}_{n-1 \text{ times}} \times \text{P}(H) \\
&= \text{P}(\bar{H})^{n-1} \times \text{P}(\bar{D})^{n-1} \times \text{P}(H) = \left(1 - \frac{1}{2}\right)^{n-1} \left(1 - \frac{1}{3}\right)^{n-1} \left(\frac{1}{2}\right) = \frac{1}{2 \times 3^{n-1}}
\end{aligned}
$$

Now let $\mathcal{H}$ be the event of stopping on a heads. Well, this is the probability of rolling a heads on the first go, or the second, or the third, and so on.

$$\begin{aligned} \mathrm{P}(\mathcal{H}) &= \mathrm{P}(H_1 \cup H_2 \cup H_3 \cup \cdots) \\ &= \mathrm{P}\left(\bigcup_{r=1}^{\infty} H_r\right) \\ &= \sum_{r=1}^{\infty} \mathrm{P}(H_r) \qquad \text{(mutually exclusive)} \\ &= \frac{1}{2}\sum_{r=1}^{\infty} \frac{1}{3^{r-1}} \\ &= \frac{1}{2}\left(\frac{1}{1-\frac{1}{3}}\right) = \boxed{\frac{3}{4}} \end{aligned}$$

[3, 3, 4 marks]

9. (a) $\left|\dfrac{z+3i-2}{z+2i+1}\right| > 2$

$$\begin{aligned} &\implies \frac{|z+3i-2|}{|z+2i+1|} > 2 \qquad \text{(by question 4(b))} \\ &\implies |z+3i-2| > 2|z+2i+1| \\ &\implies |x-2+(y+3)i| > 2|x+1+(y+2)i| \qquad \text{where } z = x+iy \\ &\implies \sqrt{(x-2)^2+(y+3)^2} > 2\sqrt{(x+1)^2+(y+2)^2} \\ &\implies (x-2)^2+(y+3)^2 > 4(x+1)^2+4(y+2)^2 \\ &\implies x^2-4x+4+y^2+6y+9 > 4x^2+8x+4+4y^2+16y^2+16 \\ &\implies 3x^2+3y^2+12x+10y+7 < 0 \\ &\implies x^2+y^2+4x+\frac{10}{3}y+\frac{7}{3} < 0 \\ &\implies (x+2)^2-4+\left(y+\frac{5}{3}\right)-\frac{25}{9}+\frac{7}{3} < 0 \\ &\implies (x+2)^2+\left(y+\frac{5}{3}\right) < \frac{40}{9} \\ &\implies (x+2)^2+\left(y+\frac{5}{3}\right) < \left(\frac{2\sqrt{10}}{3}\right)^2 \end{aligned}$$

Therefore, by comparison to the inequality $(x-a)^2+(y-b)^2 < r^2$ representing a disk centred at (a,b) with radius r, we see that points satisfying the above inequality trace a disk in the complex plane, with $\boxed{\text{centre } \left(-2, -\frac{5}{3}\right) \text{ and radius } \frac{2\sqrt{10}}{3}}$.

(b) (i) $\vec{AB} = \vec{OB} - \vec{OA} = 4\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}$.

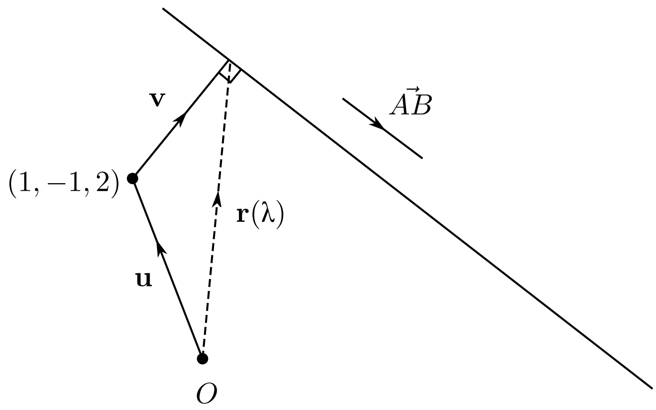
$\vec{BC} = \vec{OC} - \vec{OB} = -6\mathbf{i} + 3\mathbf{j} + 11\mathbf{k}$.

Now, consider $\vec{AB} \cdot \vec{BC} = (4\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}) \cdot (-6\mathbf{i} + 3\mathbf{j} + 11\mathbf{k}) = -24 - 9 + 33 = 0$.

Thus $\vec{AB}$ and $\vec{BC}$ are perpendicular. $\square$

(ii) $\mathbf{r}(\lambda) = A + \lambda(\vec{AB}) = \mathbf{i} + 2\mathbf{j} - \mathbf{k} + \lambda(4\mathbf{i} - 3\mathbf{j} + 3\mathbf{k})$. (alternatively, take $\mathbf{r}(\lambda) = B + \lambda(\vec{AB})$)

(iii) Let $\mathbf{u} = (1, -1, 2)$, and let the vector $\mathbf{v}$ be the vector taking us from the point $(1, -1, 2)$ to the line $\mathbf{r}(\lambda)$. We are interested in the shortest (i.e. perpendicular) distance from the given point to the line, so we want $\mathbf{v} \cdot \vec{AB} = 0$ (since $\mathbf{v}$ is perpendicular to the line, then the dot product with its direction is zero).



From the diagram, $\mathbf{r}(\lambda) = \mathbf{u} + \mathbf{v}$, so $\mathbf{v} = \mathbf{r}(\lambda) - \mathbf{u}$. Thus

$$\begin{aligned}\mathbf{v} &= \mathbf{r}(\lambda) - (\mathbf{i} - \mathbf{j} + 2\mathbf{k}) \\ &= \mathbf{i} + 2\mathbf{j} - \mathbf{k} + \lambda(4\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}) - (\mathbf{i} - \mathbf{j} + 2\mathbf{k}) \\ &= 4\lambda\mathbf{i} + (3 - 3\lambda)\mathbf{j} + (3\lambda - 3)\mathbf{k}, \qquad (*)\end{aligned}$$

and since it is desired that $\mathbf{v} \cdot \vec{AB} = 0$,

$$\begin{aligned}& (4\lambda\mathbf{i} + (3 - 3\lambda)\mathbf{j} + (3\lambda - 3)\mathbf{k}) \cdot (4\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}) = 0 \\ \Longrightarrow\ & 16\lambda + 9\lambda - 9 + 9\lambda - 9 = 0 \\ \Longrightarrow\ & 34\lambda = 18 \\ \Longrightarrow\ & \lambda = \frac{9}{17}\end{aligned}$$

Thus substituting $\lambda = \frac{9}{17}$ in $(*)$ gives us the shortest possible vector $\mathbf{v}$:

$$\mathbf{v}\big|_{\lambda=\frac{9}{17}} = \frac{36}{17}\mathbf{i} + \frac{24}{17}\mathbf{j} - \frac{24}{17}\mathbf{k},$$

thus the length $|\mathbf{v}| = \sqrt{\frac{36}{17}^2 + \frac{24}{17}^2 + \frac{24}{17}^2} = \boxed{\frac{12\sqrt{17}}{17} \text{ units}}$ is the distance from $(1, -1, 2)$ to the line $\mathbf{r}(\lambda)$.

[4, 6 marks]

10. (a) (i) $\mathbf{P}(k)$ is singular $\Longrightarrow \det \mathbf{P}(k) = 0$

$$\begin{aligned}& \Longrightarrow \begin{vmatrix} k & 2 \\ k-6 & k-5 \end{vmatrix} = 0 \\ & \Longrightarrow k(k-5) - 2(k-6) = 0 \\ & \Longrightarrow k^2 - 7k + 12 = 0 \\ & \Longrightarrow (k-3)(k-4) = 0\end{aligned}$$

$\Longrightarrow k = 3 \quad \text{or} \quad k = 4$

Thus $\mathbf{P}(k)$ is singular when $\boxed{k = 3 \text{ or } k = 4}$.

(ii) $\mathbf{P}^{-1}(k) = \dfrac{1}{\det \mathbf{P}(k)} \begin{pmatrix} k-5 & -2 \\ 6-k & k \end{pmatrix} = \boxed{\dfrac{1}{(k-3)(k-4)} \begin{pmatrix} k-5 & -2 \\ 6-k & k \end{pmatrix}}$.

(iii) For intersection, we solve the equations simultaneously. The given system of equations corresponds to the matrix equation

$$\mathbf{P}(k)\,\mathbf{x} = \mathbf{y}$$

where $\mathbf{x} = (x, y)$ and $\mathbf{y} = (12, 10)$. Thus the solution is given by

$$\mathbf{x} = \mathbf{P}^{-1}(k)\,\mathbf{y} = \frac{1}{(k-3)(k-4)} \begin{pmatrix} k-5 & -2 \\ 6-k & k \end{pmatrix} \begin{pmatrix} 12 \\ 10 \end{pmatrix} = \frac{1}{(k-3)(k-4)} \begin{pmatrix} 12k-80 \\ 72-2k \end{pmatrix},$$

i.e. ℓ_1 and ℓ_2 intersect at the point $\boxed{\left(\dfrac{12k-80}{(k-3)(k-4)}, \dfrac{72-2k}{(k-3)(k-4)} \right)}$.

If k takes on any of the values in part (a), then the matrix is singular, so it is not invertible and the system has no solutions. This can only mean that ℓ_1 and ℓ_2 are parallel.

(iv) $\begin{pmatrix} 1 \\ 10 \end{pmatrix} \xmapsto{\mathbf{P}(k)} \begin{pmatrix} 5k \\ -1 \end{pmatrix} \implies \mathbf{P}(k) \begin{pmatrix} 1 \\ 10 \end{pmatrix} = \begin{pmatrix} 5k \\ -1 \end{pmatrix}$

$$\implies \mathbf{P}^{-1}(k) \begin{pmatrix} 5k \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 10 \end{pmatrix}$$
$$\implies \frac{1}{(k-3)(k-4)} \begin{pmatrix} k-5 & -2 \\ 6-k & k \end{pmatrix} \begin{pmatrix} 5k \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 10 \end{pmatrix}$$
$$\implies \begin{pmatrix} k-5 & -2 \\ 6-k & k \end{pmatrix} \begin{pmatrix} 5k \\ -1 \end{pmatrix} = (k-3)(k-4) \begin{pmatrix} 1 \\ 10 \end{pmatrix}$$
$$\implies \begin{pmatrix} 5k^2-25k+2 \\ 30k-5k^{-}k \end{pmatrix} = \begin{pmatrix} k^2-7k+12 \\ 10k^2-20k+120 \end{pmatrix}$$
$$\implies \begin{cases} 4k^2-18k-10=0 & (1) \\ 15k^2-99k+120=0 & (2) \end{cases}$$

(1) $\implies 2k^2-9k-5=0 \implies (2k+1)(k-5)=0 \implies k=-\frac{1}{2}$ or $k=5$.

(2) $\implies 5k^2-33k+40=0 \implies (5k-8)(k-5)=0 \implies k=\frac{8}{5}$ or $k=5$.

Since both (1) and (2) must be satisfied, the only valid value of k is $\boxed{k=5}$.

(b) (i) We have $a=9$, and $a+ar+ar^2=19$.

$$\implies 9+9r+9r^2=19$$
$$\implies 9r^2+9r-10=0$$
$$\implies (3r-2)(3r+5)=0$$
$$\implies r_a=\frac{2}{3}, \quad r_b=-\frac{5}{3}$$

Thus $\boxed{a_n = 9\left(\frac{2}{3}\right)^{n-1}, \text{ and } b_n = 9\left(-\frac{5}{3}\right)^{n-1}}$.

(ii) $$\sum_{r=1}^{n} a_r = \sum_{r=1}^{n} 9\left(\frac{2}{3}\right)^{r-1} = \sum_{r=0}^{n-1} 9\left(\frac{2}{3}\right)^{r} = \frac{9(1-\frac{2}{3}^{n})}{1-\frac{2}{3}} = \boxed{27\left(1-\left(\frac{2}{3}\right)^{n}\right)}$$

$$\sum_{r=1}^{n} b_r = \sum_{r=1}^{n} 9\left(-\frac{5}{3}\right)^{r-1} = \sum_{r=0}^{n-1} 9\left(-\frac{5}{3}\right)^{r} = \frac{9(1-(-\frac{5}{3})^{n})}{1-(-\frac{5}{3})} = \boxed{\frac{27}{8}\left(1-\left(-\frac{5}{3}\right)^{n}\right)}$$

(iii) $\sum^{\infty} a_n$ converges, since $|r_a| < 1$.

$$\sum_{r=1}^{\infty} a_r = \lim_{n\to\infty}\sum_{r=1}^{n} a_r = \lim_{n\to\infty}\left[27\left(1-\left(\frac{2}{3}\right)^{n}\right)\right] = \boxed{27}\,.$$

[5, 5 marks]