Pure Mathematics
Paper I

17th April 2016

Question Paper

This paper consists of four pages and ten questions. Check to see if any pages are missing.

Answer **ALL** questions. Each question carries **10** marks.

- Protractors and scientific calculators are permitted
- Graphical calculators are **not** permitted
- Check answers fully and present working fully as necessary
- Three hours are allocated for this test paper, utilise your time effectively

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Question Paper

1. (a) Resolve the function $\dfrac{3x^2 - 2x + 11}{(x^2+1)(x-4)}$ into partial fractions.

   (b) Solve the differential equation

$$(x^2+1)(x-4)\,\frac{\mathrm{d}y}{\mathrm{d}x} = \sec^4 y\,(3x^2 - 2x + 11)$$

   given that $y = 0$ when $x = 5$.

[3, 7 marks]

2. (a) Derive the equation of the circle $\mathscr{C}$, centred on the x-axis, whose tangent at the point $(2, 1)$ is given by the equation $\ell_1 : 2y = x$.

   (b) Verify that $\ell_2 : 2y = 5 - x$ is also a tangent to the circle.

   (c) Find the point of intersection of ℓ_1 and ℓ_2, and verify (for this case) the classical geometry theorem which states that *tangents to a circle from the same point are equal in length.*

[5, 2, 3 marks]

3. (a) (i) Express $f(x) \equiv \cos x + \sqrt{3}\sin x$ in the form

$$f(x) \equiv \lambda\cos(x - \alpha)$$

   where $\lambda > 0$ and $\alpha \in \left[0, \frac{\pi}{2}\right]$.

   (ii) Sketch $y = f(x)$ in the range $[0, 2\pi]$, clearly indicating the curve's amplitude, and the points where it intersects the coordinate axes.

   (iii) Determine $\min \dfrac{1}{f(x)+1}$ and the value of $x \in [0, 2\pi]$ for which this minimum occurs.

   (b) Determine the five solutions to the equation

$$\sin\theta + \sin 3\theta + \sin 5\theta = 0$$

   for θ in the range $0 < \theta < 360°$.

[6, 4 marks]

4. (a) Determine the 2×2 transformation matrix $\mathbf{I}_*$ which rotates vectors in the plane by $90°$ anti-clockwise. Prove that:

   (i) ${\mathbf{I}_*}^2 = -\mathbf{I}$, where $\mathbf{I}$ is the 2×2 identity matrix.

   (ii) $(\mathbf{I}\cos\theta + \mathbf{I}_*\sin\theta)^n = \mathbf{I}\cos n\theta + \mathbf{I}_*\sin n\theta$, by induction on n.

   (b) Prove that for any two complex numbers z and w,

$$\left|\frac{z}{w}\right| = \frac{|z|}{|w|} \qquad \text{and} \qquad \arg zw = \arg z + \arg w$$

   How does multiplication by the imaginary number i affect complex numbers on an Argand diagram? Does this explain anything about the behaviour of $\mathbf{I}_*$ in part (a)?

[6, 4 marks]

5. (a) Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x and y for each of the following.

   (i) $x = e^t \cos t$, $y = e^t \sin t$, where $t \in \mathbb{R}$ is a parameter.

   (ii) $4x^2 - 3xy + 5y^3 + 2\cos y \sin x = 7$.

   (b) Find the tangent to the curve $y = \exp(\sin^2 x - \cos x)$ at the point where $x = \frac{\pi}{2}$.
   [Note: $\exp x \equiv e^x$]

   (c) If x and y are nonnegative real numbers such that $x+y = 15$, what is the maximum possible value of xy^2?

[4, 3, 3 marks]

6. (a) (i) Express $p(x) = 3 - 2x - x^2$ in the form $a - (x+b)^2$.

   (ii) Use the substitution $x + 1 = 2\sin u$ to show that

$$\int_0^1 \sqrt{3 - 2x - x^2}\,\mathrm{d}x = \frac{4\pi - 3\sqrt{3}}{6}$$

   (b) Determine the integral

$$\int e^{\alpha\theta} \cos \beta\theta \,\mathrm{d}\theta$$

[5, 5 marks]

7. The real-valued function f is given by

$$f(x) = \begin{cases} 1 - x & \text{for } x \leq -2 \\ a - x - x^2 & \text{for } -2 \leq x \leq 2 \\ \ln(x+b) & \text{for } x \geq 2 \end{cases}$$

   (a) Determine the constants a and b so that f is a continuous, well-defined function (i.e. the endpoints of each piece of the function meet at the same point).

   (b) Sketch f in the range $-5 < x < 5$, clearly indicating the intercepts with the coordinate axes, and state its domain and range.

   (c) The real-valued function g is given by $g(x) = 3 - |x|$ over the domain $-3 < x < 1$. Sketch the function $g(x)$ and state its range. Also, show that there is no point where $f(x) = g(x)$.

[3, 3, 4 marks]

8. (a) How many four letter words, not necessarily meaningful, can be formed out of the letters in the word

PRESIDENT

   (b) How many factors does $7! = 5040$ have?

   (c) Donald alternates between tossing a fair coin and throwing a fair dice, stopping only when he gets a head on the coin or a 4 or 6 on the dice. What is the probability that he ends when tossing a coin?

[3, 3, 4 marks]

9. (a) Show that the complex numbers satisfying $\left|\dfrac{z+3i-2}{z+2i+1}\right| > 2$ trace a disk on an Argand diagram. Find its centre and radius.

(b) Consider the positions in $\mathbb{R}^3$, given by

$$A = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \qquad B = \begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix}, \qquad \text{and} \qquad C = \begin{pmatrix} -1 \\ 2 \\ 13 \end{pmatrix}$$

(i) Determine the vectors $\vec{AB}$ and $\vec{BC}$, and show that they are perpendicular.

(ii) Determine the vector equation $\mathbf{r}(\lambda)$ of the line through the positions A and B.

(iii) Find the distance of the line $\mathbf{r}(\lambda)$ from the position $(1, -1, 2)$.

[4, 6 marks]

10. (a) A matrix $\mathbf{P}(k)$ is given by

$$\mathbf{P}(k) = \begin{pmatrix} k & 2 \\ k-6 & k-5 \end{pmatrix}$$

for $k \in \mathbb{R}$.

(i) Determine the values of k for which $\mathbf{P}(k)$ is singular.

(ii) Find, in terms of k, the inverse matrix $\mathbf{P}^{-1}(k)$ for when k is not equal to any of the values found in part (i).

(iii) The general lines ℓ_1 and ℓ_2 in $\mathbb{R}^2$ are both of the form

$$\begin{cases} \ell_1 : \quad kx + 2y = 12 \\ \ell_2 : (k-6)x + (k-5)y = 10 \end{cases}$$

where $k \in \mathbb{R}$. Find, in terms of k the general point of intersection of the lines ℓ_1 and ℓ_2. What happens when k takes any of the values found in part (i)?

(iv) Find the value of k for which $\begin{pmatrix} 1 \\ 10 \end{pmatrix} \xmapsto{\mathbf{P}(k)} \begin{pmatrix} 5k \\ -1 \end{pmatrix}$.

[Note: $\mathbf{x} \xmapsto{\mathbf{P}(k)} \mathbf{y}$ is read "$\mathbf{x}$ is mapped to $\mathbf{y}$ by $\mathbf{P}(k)$"]

(b) Two geometric progressions $\langle a_n \rangle$ and $\langle b_n \rangle$, where $n \in \mathbb{N}$, each have first term 9 and sum of the first three terms equal to 19.

(i) Find expressions for a_n and b_n.

(ii) Determine the sum of the first n terms of each of the two progressions.

(iii) State which of $\langle a_n \rangle$ and $\langle b_n \rangle$ converges, and determine its sum to infinity.

[5, 5 marks]