

Ordinary Differential Equations

Pure Mathematics A-Level: Cheat Sheet

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We cover the following first/second order ordinary differential equations (ODEs). The *order* of an ODE is determined by the highest derivative present in the equation. By *ordinary*, we mean that the derivatives are of a function (y) of only one independent variable (x).

First Order ODEs

1. Separable

These are equations which can be brought to the form

$$f(y) \frac{dy}{dx} = g(x).$$

Integrating both sides with respect to x gives the general solution.

2. Product Rule Equations

These are equations which are the direct result of applying the product rule. In general, they have the form

$$f(y) g'(x) + f'(y) g(x) = h(x),$$

which can be transformed to

$$\frac{d}{dx}(fg) = h(x).$$

Integrating both sides gives the solution.

3. Linear Equations

First order equations of the form

$$\frac{dy}{dx} + f(x)y = g(x)$$

are said to be *linear*. They can be reduced to product rule equations by multiplying throughout by

$$\mu(x) = \exp\left(\int f(x) dx\right),$$

known as the *integrating factor* (where $\exp(x) \stackrel{\text{def}}{=} e^x$).

Second Order ODEs

1. Homogeneous with Constant Coefficients

A second order ODE with constant coefficients is *homogeneous* when it equals zero. In other words, we consider the equation

$$a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + cy = 0$$

where $a, b, c \in \mathbb{R}$ are constants. First, we solve the **auxiliary equation**

$$ak^2 + bk + c = 0$$

whose solutions are $k = k_1$ and $k = k_2$. The general solution is then given by

$$y(x) = \begin{cases} c_1 e^{k_1 x} + c_2 e^{k_2 x} & \text{if } k_1 \neq k_2 \\ e^{kx}(c_1 + c_2 x) & \text{if } k = k_1 = k_2 \\ e^{\alpha x}(c_1 \sin \beta x + c_2 \cos \beta x) & \text{if } k = \alpha \pm \beta i \in \mathbb{C} \end{cases}$$

where c_1, c_2 are arbitrary constants.

2. Inhomogeneous with Constant Coefficients

A differential equation is *inhomogeneous* if it is not homogeneous. Here we consider the equation

$$a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + cy = f(x) \neq 0$$

where $a, b, c \in \mathbb{R}$ are constants. We solve by following these steps:

- (i) Solve the **homogeneous equation**

$$a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + cy = 0$$

to obtain the **complementary function** $cf(x)$.

- (ii) Guess a **trial solution**, i.e. a function $ts(x)$ which, when substituted in the left-hand side of the equation, is likely to result in $f(x)$. [Table 1](#) suggests trial solutions for common elementary functions f . Note that even if some of the constants a, b, \dots in f are zero, the corresponding constants λ, μ, \dots in the trial solution should not be assumed zero. For example, if $f(x) = x^2$, then we still take the trial solution to be $\lambda x^2 + \mu x + \eta$. Similarly, the function $f(x) = x^2 + \sin 2x$ has trial solution $\lambda x^2 + \mu x + \eta + \vartheta \cos 2x + \varphi \sin 2x$.

- (iii) Determine the trial solution derivatives $ts'(x)$ and $ts''(x)$, and substitute them in the original ODE. Compare coefficients to determine correct values for the constants so that the result will equal $f(x)$.

The trial solution with the constant(s) found is called the **particular integral**, $pi(x)$.

- (iv) The general solution is given by $y(x) = cf(x) + pi(x)$.

	$f(x)$	Trial Solution, $ts(x)$
Polynomials	a	λ
	$ax + b$	$\lambda x + \mu$
	$ax^2 + bx + c$	$\lambda x^2 + \mu x + \eta$
	\vdots	\vdots
Exponentials [†]	$ae^{\alpha x}$	$\lambda e^{\alpha x}$ if $k_1 \neq \alpha \neq k_2$
		$\lambda x e^{\alpha x}$ if $k_1 = \alpha \neq k_2$
		$\lambda x^2 e^{\alpha x}$ if $k_1 = \alpha = k_2$
Trigonometric	$a \cos \alpha x + b \sin \alpha x$	$\lambda \cos \alpha x + \mu \sin \alpha x$

[†] Note that k_1, k_2 are the solutions to the auxiliary equation solved in part (i).

TABLE 1: Trial solutions of common elementary functions.

Remark: Why does this method work?

Differentiation is an operator, that is, a function whose inputs and outputs are themselves functions. If we denote the differentiation of f by $D[f]$, then both f and $D[f]$ are functions, which when evaluated at x , yield the numbers $f(x)$ and $D[f](x)$ respectively. The symbol D alone denotes differentiation as a function in its own right. (In Leibnitz notation, this is the difference between $\frac{dy}{dx}$, which is the function $D[f]$ whose inputs are numbers, and $\frac{d}{dx}$, which is equivalent to D and whose inputs are functions.)

In general, an operator L is *linear* if for any two functions f and g ,

$$L[f + g] = L[f] + L[g] \quad \text{and} \quad L[\alpha f] = \alpha L[f],$$

where α is any constant. Indeed, the differential operator D is linear, e.g. if for all x , f and g are defined by $f(x) = \sin x$ and $g(x) = x^2$, then

$$D[2f + 3g](x) = 2 \cos x + 6x = 2D[f](x) + 3D[g](x),$$

i.e. $D[2f + 3g] = 2D[f] + 3D[g]$.

Studying linear operators abstractly is useful. Let $\mathbf{0}$ denote the zero function, i.e. the function defined by $\mathbf{0}(x) = 0$ for all x . Note that this is different from zero; the former is a function, the latter is a number. Now if L is a linear operator, the set of functions which are mapped to $\mathbf{0}$ by L is called the *kernel*, denoted $\ker L$. In other words,

$$f \in \ker L \iff L[f] = \mathbf{0}.$$

The function $\mathbf{0}$ itself is in the kernel of any linear operator L . Indeed, since for any function f , we have $(0f)(x) = 0f(x) = 0 = \mathbf{0}(x)$ for all x , then $0f = \mathbf{0}$. Hence since L is linear,

$$L[\mathbf{0}] = L[0\mathbf{0}] = 0L[\mathbf{0}] = \mathbf{0},$$

so $\mathbf{0} \in \ker L$.

The kernel of a linear operator L can tell us a lot about it, such as whether or not L is invertible. Recall that in general, a function F has an inverse if and only if it is one-to-one, i.e. if for all x and y , $F(x) = F(y)$ implies that $x = y$. Applying this reasoning to linear operators, it is easy to see that for L to have an inverse L^{-1} , only $\mathbf{0}$ must be in its kernel. Indeed, $\mathbf{0} \in \ker L$ for any L by the argument above; but if $f \in \ker L$ where $f \neq \mathbf{0}$, then by definition of $\ker L$, we have $L[f] = \mathbf{0} = L[\mathbf{0}]$, but $f \neq \mathbf{0}$. This contradicts the definition of one-to-one.

What is the the kernel $\ker D$ of the differentiation operator D ? By now, we know that $\ker D$ is precisely the set of *constant functions*, such as the function $\mathbf{3}$ where $\mathbf{3}(x) = 3$ for all x .

Now we finally address the problem of solving differential equations. The simplest differential equation is the implicit one in the evaluation of an indefinite integral $\int f(x) dx$, since this is equivalent to finding a solution $y(x)$ for the differential equation

$$\frac{dy}{dx} = f(x);$$

or with the operator notation, $D[y] = f$. Now if D were invertible, the solution would simply be $y = D^{-1}[f]$, but unfortunately the situation is not as simple, since as we have just seen, $\ker D \neq \{\mathbf{0}\}$. So how do we solve this problem? What we usually do is determine a particular function y_p by the techniques of integration, and then write

$$\int f(x) dx = y_p + c$$

where c is an “arbitrary constant”. The addition of this constants incorporates *all* solutions to the differential equation. In view of the theory of kernels we have developed, this is equivalent to doing $y_p + \mathbf{c}$ for any constant function $\mathbf{c} \in \ker D$. Indeed, if y_p is a solution, it makes sense that $y_p + \mathbf{c}$ is also a solution, since

$$D[y_p + \mathbf{c}] = D[y_p] + D[\mathbf{c}] = D[y_p] + \mathbf{0} = D[y_p].$$

But how does this incorporate *all* solutions? Say we want to solve $L[y] = f$ for any linear operator L . Let y_p be a particular solution we found somehow, and let y represent any other solution. By linearity,

$$L[y - y_p] = L[y] - L[y_p] = f - f = \mathbf{0},$$

so $y - y_p \in \ker L$, i.e. $y - y_p = k$ for some function $k \in \ker L$. Thus

$$y = y_p + k.$$

Hence we have shown that any solution y to the equation $L[y] = f$ can be written as the particular solution y_p plus some member of the kernel, and it follows that all solutions are given by $y = y_p + k$ for $k \in \ker L$.

Essentially, this is what the method described is doing. Instead of simply having $\frac{dy}{dx} = f$ though, we have equations of the form

$$a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + cy = f,$$

where $a, b, c \in \mathbb{R}$. It is easy to see that the left-hand side is also a linear operator, since it inherits linearity from the operators $\frac{d^2}{dx^2}$, $\frac{d}{dx}$ and the identity ($I[y] = y$). indeed, if we define $L[y] = a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + cy$, then

$$\begin{aligned} L[f + g] &= a \frac{d^2}{dx^2}(f + g) + b \frac{d}{dx}(f + g) + c(f + g) \\ &= a \frac{d^2 f}{dx^2} + b \frac{d^2 f}{dx^2} + cf + a \frac{d^2 g}{dx^2} + b \frac{d^2 g}{dx^2} + cg \\ &= L[f] + L[g], \end{aligned}$$

and

$$L[\alpha f] = a \frac{d^2}{dx^2}(\alpha f) + b \frac{d}{dx}(\alpha f) + c(\alpha f) = \alpha \left(a \frac{d^2 f}{dx^2} + b \frac{d^2 f}{dx^2} + cf \right) = \alpha L[f].$$

When defining such operators, we sometimes abuse notation slightly and write $L = a \frac{d^2}{dx^2} + b \frac{d}{dx} + c$ or $L = aD^2 + bD + cI$ instead of $L[y] = a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + cy$ for all y . Since the given equation is equivalent to $L[y] = f$, we may also write

$$\left(a \frac{d^2}{dx^2} + b \frac{d}{dx} + c \right) [y] = f \quad \text{or} \quad (aD^2 + bD + cI)[y] = f.$$

Let's take an example, say,

$$\frac{d^2 y}{dx^2} - 5 \frac{dy}{dx} + 6y = \cos 2x.$$

In this case, we have the operator $L = D^2 - 5D + 6I$. The first thing we need to do is to study this operator L , in particular, we need to find its kernel. In general, operators of the form $aD^2 + bD + cI$ have an exponential function $f(x) = e^{kx}$ in their kernel for some value of k .

Indeed, since $D[f](x) = ke^{kx}$ and $D^2[f](x) = k^2 e^{kx}$, then

$$L[f](x) = ak^2 e^{kx} + bke^{kx} + ce^{kx} = e^{kx}(ak^2 + bk + c).$$

Since $e^{kx} \neq 0$ for all $x \in \mathbb{R}$ (or \mathbb{C}), it follows that $L[f] = \mathbf{0}$ whenever k is a solution to the auxiliary equation $ak^2 + bk + c = 0$. There are some technical details as to why we take different general solutions depending on the multiplicity of k , or whether it is real or complex, but essentially, the first step of the solution process is determining the kernel $\ker L$ of the linear operator defined by the left-hand side. This is what the complementary function achieves.

The trial solution part of the method is effectively just a guess for a particular solution (hence its name), y_p . Once a correct solution is found, the general solution is given by $y = y_p + k$, as described in the general framework of linear operators.

And that's why it works.