Induction

Pure Mathematics A-Level

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1 Introduction

All mathematical proofs done so far have been **deductive**, where one step is deduced from the other (Step 1 $\Rightarrow$ Step 2 $\Rightarrow \cdots \Rightarrow$ Step n). A relatively newer concept (for which we should thank Pascal) is mathematical proof by **induction**. Induction is a powerful tool we use to prove statements for all natural numbers ($\mathbb{N}$ or $\mathbb{Z}^+$), or for any discrete structure in general.

The idea is based on what we often call the *domino effect*. Consider an (infinite) line of dominoes. We wish to prove that all the dominoes will fall, and we do this as follows:

(1) Prove that the first one will fall.

(2) Prove that if the kth domino falls, then the $k+1$st domino falls also.

Suppose we manage to prove both (1) and (2). By (1), we know that the first domino falls. But then by (2), the second one falls also. But now we can use (2) again to conclude that the third domino falls. And the fourth. And the fifth. This can continue indefinitely, thus proving that all the dominoes fall.

Let us transfer this to a mathematical context. Suppose we wish to prove that a statement S_n is true $\forall\, n \in \mathbb{N}$. We can do this by:

(1) Proving that S_n is true for the case $n = 1$, i.e. proving that S_1 is true. This is called the **inductive basis** or the **base case**.

(2) Assume that S_n is true for the case $n = k$ (we call this the **inductive hypothesis**). Based on this assumption, prove that S_n is true for the case $n = k + 1$. This is called the **inductive step**.

2 Examples

We give an example of induction on series. Suppose we wish to prove the result $1 + 2 + \cdots + n = \frac{n}{2}(n+1)$ for all $n \in \mathbb{N}$.

Proof. Base case: For $n = 1$, we have $1 = \frac{1}{2}(1+1)$, so the statement holds.

Assume the result holds for $n = k$, i.e. that $1 + 2 + \cdots + k = \frac{k}{2}(k+1)$. Can we show that it is true for $n = k + 1$?

$$\begin{aligned} 1 + 2 + \cdots + k + (k+1) &= \frac{k}{2}(k+1) + k + 1 \quad \text{(by induction hypothesis)} \\ &= \frac{k+1}{2}(k+2) \\ &= \frac{k+1}{2}((k+1)+1) \end{aligned}$$

i.e. the statement holds for $n = k + 1$. Therefore by the principle of induction, the statement holds $\forall\, n \in \mathbb{N}$. □

Another example: induction on divisibility. We show that the number $4^n + 2$ is always divisible by 3.

Proof. Base case: When $n = 1$, we have $4^1 + 2 = 6 = 3(2)$, so the statement holds.

Assume the result holds for $n = k$, i.e. that $4^k + 2 = 3\alpha$ for some $\alpha \in \mathbb{N}$. Can we show that it is true for $n = k + 1$?

$$\begin{aligned} 4^{k+1} + 2 &= 4(4^k) + 2 \\ &= 4(4^k) + 4 - 2 \\ &= 4(4^k + 1) - 2 \\ &= 4(4^k + 2 - 1) - 2 \\ &= 4(3\alpha - 1) - 2 \quad \text{(by induction hypothesis)} \\ &= 12\alpha - 4 - 2 \\ &= 12\alpha - 6 \\ &= 3(4\alpha - 2) \\ &= 3\beta \qquad \text{where } \beta = (4\alpha - 2) \in \mathbb{N} \quad \forall \alpha \in \mathbb{N} \end{aligned}$$

i.e. the statement holds for $n = k + 1$. Therefore by the principle of induction, the statement holds $\forall\, n \in \mathbb{N}$. □

A final example with matrices. We prove that if $\mathbf{A} = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$, then the matrix power $\mathbf{A}^n$ is given by $\mathbf{A}^n = \begin{pmatrix} 3^n & 3^n - 2^n \\ 0 & 2^n \end{pmatrix}$.

Proof. Base case: $\mathbf{A}^1 = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 3^1 & 3^1 - 2^1 \\ 0 & 2^1 \end{pmatrix}$, so the statement holds.

Assume the result holds for $n = k$, i.e. that $\mathbf{A}^k = \begin{pmatrix} 3^k & 3^k - 2^k \\ 0 & 2^k \end{pmatrix}$. Can we show that it is true for $n = k + 1$?

$$\begin{aligned}
\mathbf{A}^{k+1} &= \mathbf{A}^k \mathbf{A} \\
&= \begin{pmatrix} 3^k & 3^k - 2^k \\ 0 & 2^k \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix} \\
&= \begin{pmatrix} 3^k(3) + 0 & 3^k + 2(3^k - 2^k) \\ 0 & 2^k(2) \end{pmatrix} \\
&= \begin{pmatrix} 3^{k+1} & 3^k + 2(3^k) - 2^{k+1} \\ 0 & 2^{k+1} \end{pmatrix} \\
&= \begin{pmatrix} 3^{k+1} & 3^k(1+2) - 2^{k+1} \\ 0 & 2^{k+1} \end{pmatrix} \\
&= \begin{pmatrix} 3^{k+1} & 3^{k+1} - 2^{k+1} \\ 0 & 2^{k+1} \end{pmatrix}
\end{aligned}$$

i.e. the statement holds for $n = k + 1$. Therefore by the principle of induction, the statement holds $\forall\, n \in \mathbb{N}$. □

3 Remarks

Observe that the challenge in proof by induction lies in the inductive step. In most cases, we aim to decompose the statement S_{k+1} into S_k and other components, so that the hypothesis can then be substituted for S_k.

This powerful method opens many doors into proof writing and can make the most complicated theorems about the integers trivial to prove. Notice how we can prove statements about series, numbers, matrices, and all kinds of discrete constructs.