Coordinate Geometry

Pure Mathematics A-Level

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The following is a list of definitions and results which are useful in solving classical geometry problems. Note that the symbol Δ in front of a variable denotes the change in that variable; for example if x takes on two values x_0 and x_1, then $\Delta x = |x_0 - x_1|$.

1. The **distance** between two points $A = (x_0, y_0)$ and $B = (x_1, y_1)$, denoted $d(A, B)$, is defined
$$d(A, B) := \sqrt{\Delta x^2 + \Delta y^2}$$
which, without Δ-notation, is $d(A, B) = \sqrt{(x_0 - x_1)^2 + (y_0 - y_1)^2}$.

2. The **midpoint** of the line segment joining the points $A = (x_0, y_0)$ and $B = (x_1, y_1)$ is given by
$$M = \left(\frac{x_0 + x_1}{2}, \frac{y_0 + y_1}{2}\right),$$
i.e. the *average* of the two points. One can verify that this point M satisfies the desired properties; namely:
   - $d(A, M) = d(M, B)$, i.e. it lies in the middle of A and B, and
   - M lies on the line joining A and B.

3. The **gradient** of a line $\ell \subseteq \mathbb{R}^2$, denoted m_ℓ, is a measure of how steep ℓ is, defined
$$m_\ell := \frac{\Delta y}{\Delta x}$$
for any two points $A, B \in \ell$. Observe that:
   - The gradient m_ℓ is invariant; i.e. for any two points $A, B \in \ell$ we choose, m_ℓ remains the same.
   - $m_\ell = \tan\theta$, where θ is the angle that ℓ makes with the positive x-axis.
   - If two lines ℓ_1 and ℓ_2 are *parallel*, we write $\ell_1 \parallel \ell_2$, and
$$\ell_1 \parallel \ell_2 \iff m_{\ell_1} = m_{\ell_2}.$$

- If two lines ℓ_1 and ℓ_2 are *perpendicular*, we write $\ell_1 \perp \ell_2$, and

$$\ell_1 \perp \ell_2 \iff m_{\ell_1} = -\frac{1}{m_{\ell_2}}.$$

4. The **equation of a line** $\ell \subseteq \mathbb{R}^2$ is an equation in x and y, whose satisfaction by some point (x_0, y_0) is both a *necessary and sufficient* condition for (x_0, y_0) to be a point on ℓ.

   If $A = (x_0, y_0)$ is a fixed point on ℓ and $m = m_\ell$, then the equation of the line ℓ is

   $$y - y_0 = m(x - x_0).$$

   Alternatively, taking $(0, c)$ as the fixed point on ℓ, i.e. the y-coordinate of ℓ as it crosses the y-axis, we get the equation

   $$y = mx + c.$$

5. Suppose two curves $\mathcal{C}_1, \mathcal{C}_2 \subseteq \mathbb{R}^2$ are represented by the equations $c_1(x, y) = 0$ and $c_2(x, y) = 0$ in x and y. Then any **points of intersection** of the two curves are given by solving the system of equations

   $$\begin{cases} c_1(x, y) = 0 \\ c_2(x, y) = 0 \end{cases}$$

   simultaneously. Any solution (x_0, y_0) to this system corresponds to a point $(x_0, y_0) \in \mathcal{C}_1 \cap \mathcal{C}_2$.

6. The **acute angle** θ between two lines $\ell_1, \ell_2 \subseteq \mathbb{R}^2$ with gradients $m_1 = m_{\ell_1}$ and $m_2 = m_{\ell_2}$ is given by

   $$\theta = \tan^{-1} \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|.$$

7. The **distance** between the line $\ell : ax + by + c = 0$ and the point $A = (h, k)$ is given by

   $$d(A, \ell) = \frac{|ah + bk + c|}{\sqrt{a^2 + b^2}}.$$

8. The **circle** $\mathcal{C} \subseteq \mathbb{R}^2$ with centre $C = (a, b)$ and radius r has the standard equation

   $$(x - a)^2 + (y - b)^2 = r^2.$$

   Any quadratic equation of the form $x^2 + y^2 + ax + by + c = 0$ represents a circle[1], and can be brought to the form stated above by completing the square twice (once with the variable x, once with the variable y).

   Alternatively, one can expand the equation given above and compare the coefficients.

[1] Additionally, it must satisfy the property $a^2 + b^2 > 4c$. Why do you think this is? What would the curve represent if it does not satisfy this property?